Aristotelian Diagrams for the Ancient Discussion on Privative and Infinite Negation

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Abstract

This paper is concerned with the ancient discussion on privative negation (e.g., 'unjust') and infinite negation (e.g., 'not-just'). We formalize and compare the positions of Aristotle and Alexander of Aphrodisias, of Proclus and Ammonius Hermiae, and of Porphyry (as presented by Boethius). Each of these formalizations takes the form of a logical system, which is intended to capture the main tenets of the position it formalizes. As an additional point of reference, we also discuss the system of contemporary, Boolean predicate logic. Our comparison focuses on the diagrams that each position gives rise to, and we show how our formalizations provide a unified and systematic perspective on these diagrams. In particular, we argue that the ancient discussion on privative and infinite negation can be understood through the lens of the so-called 'logic-sensitivity' of Aristotelian diagrams.

Keywords: Privative terms, infinite terms, indefinite terms, Aristotle, Alexander of Aphrodisias, Porphyry, Proclus, Ammonius, square of opposition, logical geometry, logic-sensitivity, bitstring semantics.

1. Introduction

In Chapter 10 of *De Interpretatione* (Ackrill 1975), Aristotle formulates a theory of the negation of the predicate. He compares several sets of propositions, but one of these sets in particular has caught the attention of several ancient commentators. In the passage *19b 22 – 30*, Aristotle constructs a diagram similar to the classical square of opposition. The difference is that the square he analyzes in this passage contains categorical propositions with infinite predicates, such as 'not-just'. While discussing the relations in this square, Aristotle mentions that a proposition with an *infinite* predicate (e.g., 'man is not-just') is related to the affirmative proposition (e.g., 'man is just') in the same way as the corresponding proposition with a *privative* predicate (e.g., 'man is unjust'). Faced with this suggestion, subsequent commentators attempting to interpret this passage have proposed several accounts of the precise logical relations holding among these propositions. These proposals jointly constitute what we call in this paper 'the ancient discussion on privative and infinite negation'.

The overarching goal of the paper is to critically examine the main positions in this discussion. More concretely, the proposals of Aristotle (384 – 322 BCE) and Alexander of Aphrodisias (fl. ca. 200 CE), of Proclus (ca. 412 – 485) and Ammonius Hermiae (ca. 440 – 520), and of Porphyry (ca. 234 – 305) — from the exposition of Boethius (ca. 476 – 526) — will be formally analyzed and compared with each other. Each of these formalizations takes the form of a logical system (i.e., a consequence relation $\models_C$ over a class of models C), which is intended to capture the main tenets of the position it formalizes. (As an additional point of reference, we will also discuss the system of contemporary, 'Boolean' predicate logic.) Our comparison will focus on the diagrams that were originally proposed by the ancient authors, and we will show how our formalizations provide a unified and systematic perspective on these diagrams.

The paper is structured as follows. Section 2 presents the relevant historical background regarding the ancient discussion on privative and infinite negation. Subsection 2.1 starts by discussing the syntax of Aristotle's propositions, as laid out in *De Interpretatione*. Subsection 2.2 then turns to the origin of the discussion on privative and infinite negation, viz., the proposal first put forward by Aristotle and later elaborated by Alexander of Aphrodisias. Next, Subsection 2.3 sets forth the solution by the Aristotelian commentators Proclus and his pupil Ammonius Hermiae, which is explained in Ammonius' commentary on Aristotle's *De Interpretatione*. This proposal is summarized in what we have called 'Ammonius' hexagon'. Finally, Subsection 2.4 deals with Porphyry's proposal, as analyzed and presented by Boethius in his commentary on Aristotle's *De Interpretatione*. Porphyry maintains a thesis of equivalence between privative and infinite predicates. This implies that Ammonius' hexagon collapses into a square, which we have called 'Porphyry's square'.

Section 3 presents the formal-logical notions that will be needed later in the paper. Subsection 3.1 introduces three logical systems based on the proposals of the ancient authors analyzed in Section 2, as well as the system of contemporary, 'Boolean' predicate logic. The main novelty of these systems is that next to the sentential negation $\neg$, they also contain predicate modifiers $\widehat{\ }$ and $\bar{\ }$, which are intended to capture privative and infinite negation. After explaining the language (which is shared by all logical systems under consideration), we discuss the semantics, with an emphasis on the definition of the models that characterize each system. Next, Subsection 3.2 introduces some key ingredients from logical geometry, i.e., the formal study of Aristotelian diagrams, such as the notion of Aristotelian isomorphism and the technique of bitstring semantics. It also introduces several other types of Aristotelian diagrams next to the classical square of opposition, including two kinds of hexagon of opposition.

Section 4 puts everything together, by using the formal tools introduced in Section 3 in order to critically analyze and compare the various positions laid out in Section 2. In Subsections 4.1 – 4.4, we present the Aristotelian diagrams that each position gives rise to, discuss their bitstring semantics, and examine how the various diagrams (and their respective bitstring semantics) relate to each other. In Subsection 4.5, we argue that the ancient discussion on privative and infinite negation can be understood through the contemporary lens of the so-called 'logic-sensitivity' of Aristotelian diagrams.

Finally, Section 5 wraps everything up, by summarizing the philosophical and formal results obtained in this paper. It also mentions some avenues for further research, pertaining to various areas of logic and philosophy.

2. The Ancient Discussion on Privative and Infinite Terms

2.1. *Aristotelian Syntax in De Interpretatione*

Throughout this paper we will follow the syntactic classification that Aristotle offers in *De Interpretatione*. We will therefore review the main types of propositions that Aristotle outlines in that work. We will start with the most general level of classification. Propositions can be:

A) existential. E.g., 'man is'.
B) of two terms. E.g., 'man runs'.
C) of three terms. E.g., 'man is just'.

Existential propositions are characterized by the use of the verb ἐστίν as an existential mode, unlike three-term ones, where the verb ἐστίν serves a copulative function. On the other hand, two- and three-term propositions differ by the introduction of the verb ἐστίν as an additional element of signification. Besides being a grammatical distinction, we will later see that this also has logical implications with respect to the use of negation. The first classification is subdivided as follows. Propositions of types A, B, and C can be:

a) quantified. E.g., 'every man is', 'every man runs', and 'every man is just'.
b) unquantified. E.g., 'man is', 'man runs' and 'man is just'.

The first subtype includes the traditional categorical propositions of the classical square of opposition, with universal (πᾶς) and particular (τις) quantifiers. Unquantified propositions are again subdivided into two more cases:

b1) singular. E.g., 'Socrates is', 'Socrates runs' and 'Socrates is just'.
b2) undetermined. E.g., 'man is', 'man runs' and 'man is just'.

The main characteristic of singular propositions is that they contain a singular term as subject. Singular terms, such as 'Callias', designate only one object, whereas universal terms, such as 'animal', designate multiple objects.[1] Undetermined propositions were not called like this by Aristotle; this terminology arose later.[2]

[1]"I call universal that which is by its nature predicated of a number of things, and particular that which is not; man, for instance, is a universal, Callias a particular." (Ackrill 1975, *17a 38*).

[2]Ammonius already calls them ἀπροσδιόριστος, which Blank translates as 'undetermined' (Blank 2013, p. 90, ll. 1 – 2). Undetermined propositions are part of the division that Ammonius presents with respect to the subject term. The subject of such propositions is said to be 'undetermined', since it lacks a quantifier. This characteristic renders the logical relation that holds between two opposite (affirmative and negative) undetermined propositions quite problematic. Ammonius devotes several

The classification of propositions has thus far been based on the following criteria: the quantity of the proposition, the number of terms, and the presence and function of the verb ἐστίν. The next division considers the type of terms that occur in the proposition. Recall that for Aristotle, a name is "a spoken sound significant by convention, without time, none of whose parts is significant in separation" (Ackrill 1975, *16a 19 – 20*), while a verb has two additional features, viz., it adds to its meaning a temporal reference and is always a sign of something said of something else.[3]

Each of these types of term can be divided into three classes: the simple, the infinite (ἀόριστον), and the modes or modifications (πτώσεις ὀνόματος). The first are the terms without additional modification, such as 'man', 'white', etc.; the second are formed by adding the negative particle, such as 'not-man', 'not-just', etc.; the third are obtained by declining the noun or conjugating the verb, such as 'of Philo' and 'walked', respectively. This typology of terms induces the following classification of propositions, which applies to propositions of all previous types (A, B, and C). Propositions can have:

i) an infinite predicate. E.g., 'some man is not-just'.
ii) an infinite subject. E.g., 'not-man runs'.
iii) both an infinite subject and an infinite predicate. E.g., 'every not-man is not-just'.

The final distinction concerns affirmative and negative propositions:

I) affirmative. E.g., 'every man runs', 'man is just'.
II) negative. E.g., 'not every man runs', 'man is not just'.

We conclude with a brief methodological reflection. In the remainder of this section, we will present Aristotle's theory of negation, as well as those of his ancient commentators. In Sections 4 and 5, these theories will be formalized and studied using (extensions of) first-order logic. More concretely, our objective is to explore the various ways in which privative and infinite negation can be integrated into contemporary logic, inspired by the ancient debate about these issues. We believe that our formal systems can shed new light on the ancient debate, while acknowledging that more historical and philological scholarship continues to be

passages to the explanation of this difficulty (Busse 1897, p. 110, l. 15 – p. 118, l. 29). He begins by considering the possible equivalence between undetermined and particular propositions. Subsequently, he finds a difficulty with the equivalence between the undetermined negation and the particular negation, which leads him to consider the option that the undetermined negation is equivalent to the universal negation instead. In considering this option, he alludes to the philosopher Syrianus, a possible teacher of Proclus, who in turn was the mentor of Ammonius. Finally, Ammonius considers an intermediate position, which is based on grammar, and which depends on a dynamic interpretation of negation. Boethius also discusses this question. In his *De syllogismo categorico* (Migne 1891*a*, p. 802C, l. 4 – p. 803B, l. 11), he proposes to interpret undetermined propositions as particular ones, and in his *Introductio ad syllogismos categoricos* (Migne 1891*b*, p. 776C, l. 4 – p. 778A, l. 11), he presents some arguments in favor of universal propositions entailing undetermined propositions, in the same way that universal propositions entail particular ones.

[3]Ammonius considers at least five senses in which Aristotle conceives of names (Blank 2013, p. 45, l. 7 – p. 46, l. 19).

necessary as well. However, directly addressing such scholarship is beyond the scope of the current paper.

In order to integrate privative and infinite negation into contemporary first-order logic, we need to carefully consider their semantic implications. For example, privative negation differs from simple negation in that it involves a notion of absence or deprivation. For example, in the statement 'Socrates is not mortal', the simple negation denies the predicate 'mortal' of the subject 'Socrates'. However, in the statement 'Socrates is immortal', the privative negation indicates a *deprivation* of the property, rather than a mere *denial*. It is clear that the semantics of first-order logic needs to be extended in order to capture such nuanced distinctions.

2.2. *Aristotle and Alexander of Aphrodisias*

In Chapter 10 of *De Interpretatione* (Ackrill 1975), Aristotle presents a theory of negative terms, commonly called 'metathetic', 'transposed', 'infinite' or 'indefinite'.[4] In that chapter, he describes several relations between pairs of contradictory propositions, and presents some lists of contradictory pairs that are similar to the well-known classical square of opposition. One of these arrangements has been a focus of attention during various episodes in the history of logic (Thompson 1953, Whitaker 1996, Correia 1997; 2006), and this diagram constitutes the starting point of our analysis. The arrangement is found at *De Int. 19b 22 – 30* (Ackrill 1975):

> But when 'is' is predicated additionally as a third thing, there are two ways of expressing opposition. (I mean, for example, 'a man is just'; here I say that the 'is' is a third component — whether name or verb — in the affirmation.) Because of this there will here be *four* cases (two of which will be related, as to order of sequence, to the affirmation and negation in the way the privations are, while two will not). I mean that 'is' will be added either to 'just' or to 'not-just', and so, too, will the negation. Thus there will be four cases. What is meant should be clear from the following diagram:

(a) 'a man is just'	(b) 'a man is not just' This is the negation of (a)
(d) 'a man is not not-just' This is the negation of (c)	(c) 'a man is not-just'

[4] 'Metathetic' and 'transposed' stay closest to the Greek 'ἐκ μεταθέσεως', while 'infinite' and 'indefinite' correspond to the Latin 'infinita'. In this paper, we systematically use the term 'infinite', while being aware that the most suitable translation is still under debate. For example, Correia (1997; 2006) uses 'indefinite' to translate both 'ἐκ μεταθέσεως' and 'infinita', and Blank (2013) accepts the same translation. Bäck (2011) switches back and forth between 'metathetic' and 'indefinite'. Finally, Read (2015) prefers 'infinite', while reserving 'indefinite' for propositions that we have called 'undetermined' in Subsection 2.1 (Read 2020).

$$\begin{tikzcd} A \arrow[r, dash] \arrow[d] \arrow[dr, dashed, dash] & B \arrow[dl, dotted, dash] \\ D \arrow[r, dash] & C \arrow[u] \end{tikzcd}$$

Figure 1: Aristotle's square in *Pr. An., 51b 36 – 38*. Full, dashed and dotted lines visualize contradiction, contrariety and subcontrariety, respectively; arrows visualize subalternations.

> 'Is' and 'is not' are here added to 'just' and to 'not-just'.

This passage has been so much commented on because it is not so clear why Aristotle suddenly includes privation (στέρησις) in his analysis. In the words of Manuel Correia: "The difficulty of this passage [. . .] is such that all commentators in trying to explain it have had the feeling of guessing at Aristotle's thought" (2006, p. 41). The specific problem is to determine in what way the privative propositions ('a man is unjust' and 'a man is not unjust') are related to the four propositions mentioned in the passage, and why they serve as a reference to produce the diagram.

If we look at the evidence Aristotle presents in Chapter 46 of Book I of the *Prior Analytics*, the diagram relating the propositions in the aforementioned passage would be as shown in Figure 1. In that passage, Aristotle explicitly states the following:

> Let 'to be good' be designated by A, 'not to be good' by B, 'to be not-good' by C (under B), and 'not to be not-good' by D (under A). Then one or the other of A and B will belong to everything and never both to the same; also one or the other of C and D, and never both to the same. (Striker 2009, *51b 36 – 40*)

Here, Aristotle points out that simple as well as infinite propositions come in contradictory pairs (A/B and C/D, respectively), and explains how simple and infinite propositions should be positioned relative to each other, thereby explicitly describing a diagram. He puts the simple affirmative ('to be good', A) and the simple negative ('not to be good', B) at the top. At the bottom, the infinite affirmative ('to be not-good', C) goes under the simple negative, and the infinite negative ('not to be not-good', D) under the simple affirmative. Subsequently, he mentions the following:

> And B will necessarily belong to whatever C belongs to; for if it is true to say that a thing is not-white, it will also be true that it is not white, since it is impossible to be white and not-white at the same time, or to be a not-white log and to be a white log. So if the affirmation does not belong to a thing, the denial will. But C will not always belong to B,

> for what is not a log at all will not be a not-white log either. (Striker 2009, *52a 1 – 6*)

In this passage, Aristotle explains that there is a subalternation — i.e., a one-way entailment — from the infinite affirmative (C) to the simple negative (B). Aristotle's explanation is somewhat obscure, but it becomes clearer if we consider the information that he provides earlier in the chapter. A few lines before the quoted passage, Aristotle argues that infinite affirmatives (e.g., 'it is not-white') are not equivalent to simple negatives (e.g., 'it is not white'), explicitly mentioning that "'it is a not-white log' and 'it is not a white log' do not belong to something at the same time. For if it is a not-white log it will be a log, whereas it is not necessary that what is not a white log be a log" (Striker 2009, *51b 29 – 35*). With this example, Aristotle intends to establish the distinction between infinite terms and simple negation on the basis of the existence or 'appropriateness' of a subject.[5]

Taking 'Socrates' as a subject for both predicates in question, the argumentation posits that Socrates may not be a log (thus failing to be appropriate for 'is a white log' as well as for 'is a not-white log'), in which case the infinite affirmative ('Socrates is a not-white log') is false, whereas the simple negative ('Socrates is not a white log') is true. This feature distinguishes the two propositions: they do not have the same truth conditions, and are thus not equivalent to each other. This argument also shows that the simple negative is broader (i.e., true in more circumstances) than the infinite affirmative: we can deny that Socrates is a white log (because Socrates is not a log to begin with), but we cannot affirm that Socrates is a not-white log (again, because Socrates is not a log to begin with). It is this kind of reasoning that lies behind Aristotle's justification of the subalternation from the infinite affirmative (C) to the simple negative (B). First of all, the entailment from 'it is a not-white log' to 'it is not a white log' is proven as follows:

1. 'It is a not-white log' and 'it is a white log' cannot be true at the same time.
2. Hence, if 'it is a not-white log' is true, then 'it is a white log' is false.
3. As contradictories, 'it is a white log' is false iff 'it is not a white log' is true.
4. Hence, if 'it is a not-white log' is true, then 'it is not a white log' is true.

The proof of the subalternation is completed by showing that there is no entailment from 'it is not a white log' to 'it is a not-white log':

1. Consider Socrates, who is not a log.
2. Hence, 'it is a white log' is false and 'it is a not-white log' is false.

[5]For example, for an entity to be appropriate for 'is a white log', it has to be a log to begin with. Similarly, for an entity to be appropriate for 'is a not-white log', it has to be a log to begin with. From this perspective, existence can be viewed as a kind of minimal or universal appropriateness condition: for an entity to be appropriate for any predicate whatsoever, it has to exist to begin with.

3. As contradictories, ‘it is a white log’ is false iff ‘it is not a white log’ is true.
4. Hence, ‘it is not a white log’ is true and ‘it is a not-white log’ is false.
5. This shows that there is no entailment from ‘it is not a white log’ to ‘it is a not-white log’.

The justification of subalternation between A and D is completely analogous, as we can see in the following passage:

> In reverse order, though, D belongs to everything to which A belongs. For it will be one or the other of C and D, and since it is not possible to be not-white and white at the same time, D will belong. For of what is white it is true to say that it is not not-white. But A does not belong to every D, for of what is not a log at all it is not true to say A (that it is a white log), so that D is true and A (that it is a white log) is not true. (Striker 2009, *52a 7 – 12*)

This exhaustively describes the vertical relations, i.e., the subalternations from A to D and from C to B. Aristotle also briefly mentions something about the diagonal relations:

> It is also obvious that A and C cannot belong to the same thing, while B and D can belong to the same thing. (Striker 2009, *52a 12 – 14*)

Aristotle maintains that A and C are contraries, since these predicates cannot belong to the same thing (i.e., cannot be true together). However, he remains silent about whether they can be jointly absent (i.e., can be false together), which is also involved in the definition of contrariety. Nevertheless, this condition is implicitly present after all, since A has already been argued to be contradictory to B, i.e., the subaltern of C. In a similar fashion, we can see that B and D are subcontraries: Aristotle explicitly mentions that they can belong to the same thing (i.e., can be true together), and although he remains silent about whether they cannot be jointly absent (i.e., cannot be false together), this is implicitly present in the passage after all, since B has already been argued to be contradictory to A, i.e., the superaltern of D.

Putting everything together, the passage we have discussed yields a square of opposition for simple and infinite singular propositions, as shown in Figure 2. In his commentary on the *Prior Analytics* (Mueller 2013, p. 405, l. 20 – p. 409, l. 9), Alexander of Aphrodisias agrees with this interpretation and lists the same relations that we have mentioned. In that sense, this diagram can be attributed to both authors without any problem.

Finally, and this is what connects *Pr. An. 51b 36 – 52a 14* to the passage *De Int. 19b 22 – 30* with which we started this subsection, Aristotle mentions the following:

> The privations too are similarly related to their predications in this arrangement: let 'equal' be designated by A, 'not equal' by B, 'unequal' by C, 'not unequal' by D. (Striker 2009, *52a 15 – 17*)

This passage seems to make it clear that in *De Interpretatione*, Aristotle is actually referring to *two* squares: one that combines the simple with the infinite propositions (cf. Figure 2), and a second, completely similar one that combines the simple with the privative propositions (cf. Figure 3). Unsurprisingly, the same also applies to Alexander of Aphrodisias, cf. Mueller (2013, p. 397, l. 1 – p. 418, l. 20). Let us start by quickly reviewing what Boethius says about Alexander's position:

> There is also another, simpler exposition, which Alexander expressed as follows (after many other expositions to which he directed his attention): "Since there are," he said, "four propositions, of which two are infinite and two are simple, and these two infinites are equally related to the affirmative and negative privatives, but the two simples are not similarly related to these privatives. For the affirmative infinite agrees with the affirmative privative; for that which says 'a man is not-just' agrees to the privative affirmation which says 'a man is unjust'."[6] (Meiser 1880, p. 292, ll. 8 – 18)

In this report, Boethius is explaining the position of Aristotle's commentators on the passage we quoted at the beginning of this section. The issue that all these commentators face consists in giving an accurate explanation of Aristotle's words. Alexander's position is the last one that Boethius comments on, and he offers only two passages where he summarizes his interpretation. Alexander starts, like Aristotle, from the four propositions that make up the square of simple and infinite propositions:

- a man is just
- a man is not just
- a man is not-just
- a man is not not-just

[6]Est alia quoque simplicior expositio, quam Alexander post multas alias expositiones in quibus animum vertit edidit hoc modo: cum sint, inquit, quatuor propositiones, quarum duae sunt infinitae, duae vero simplices, duae, inquit, infinitae aequaliter se habent secundum affirmationem et negationem ad privatorias, duae vero simplices ad easdem privatorias se similiter non habent hoc modo: affirmativa enim infinita consentit affirmativae privatoriae. Ea enim quae dicit infinita affirmatio est non iustus homo ei consentit privatoriae affirmationi quae dicit: Est iniustus homo. (When an ancient source is, to the best of our knowledge, not available in English translation, we have translated it ourselves, while referring to the relevant place in the critical edition and providing the original Greek/Latin text in a footnote.)

Alexander's interpretation is that the last two propositions in this list, i.e., those which have infinite predicates, are related to the following two, which have privative predicates:

- a man is unjust
- a man is not unjust

More concretely, these two pairs of propositions are said to 'agree' with each other (the term used by Alexander, according to Boethius, is *consentit*). The infinite affirmative thus 'agrees' with the privative affirmative, and the infinite negative 'agrees' with the privative negative. This agreement consists in the fact that they produce the same predicative structure: just like the infinite propositions contains a negative particle in their predicate term, so do the privative propositions contain a negative syntactic element in their predicate. In this passage, Alexander (as quoted by Boethius) does not mention anything about opposition relations, since he only distinguishes the contradictory pairs in a syntactic fashion. However, in his commentary on *Prior Analytics* (Mueller 2013), Alexander does explicitly state his agreement with Aristotle's arrangement:[7]

> And he now shows that the privative contradictory pair keeps the same position with respect to the simple pair in this arrangement and sequence as the contradictory pair by transposition does. (Mueller 2013, p. 409, ll. 22 – 24)

Alexander also adds:

> For if we place the privative contradictory pair under the simple contradictory pair, the affirmation under the negation and the negation under the affirmation, the entailment relations will be the same in arrangement as that in the case of the contradictory pair by transposition set out a little earlier. (Mueller 2013, p. 409, ll. 27 – 31)

Although this might seem to resolve all ambiguities in *De Int. 19b 22 – 30*, there remains at least one question that has not been addressed yet, viz., how are the privative and the infinite propositions related among each other?[8] Or put

[7]Alexander uses 'by transposition' ('ἐκ μεταθέσεως') for infinite propositions.

[8]Alexander of Aphrodisias does not solve this either in his commentary on the *Prior Analytics* (Mueller 2013). According to Correia (1997; 2006), Alexander's interpretation (as reported by Boethius; cf. Meiser 1880, p. 292, ll. 8 – 18) moves from the innocent claim that privative and infinite propositions are *similar*, "because both are negations of positive meaning" (Correia 1997, p. 287), to the much more wide-reaching claim that they are *semantically equivalent*, "without any proof of its validity" (Correia 1997, p. 288). In the following, we will focus exclusively on Porphyry as maintaining this more extreme position.

$$\begin{tikzcd} \textit{a man is just} \arrow[r, dash] \arrow[d] \arrow[dr, dash, dashed] & \textit{a man is not just} \\ \textit{a man is not not-just} \arrow[r, dash] \arrow[ur, dash, dotted] & \textit{a man is not-just} \arrow[u] \end{tikzcd}$$

Figure 2: Square of opposition for infinite propositions.

$$\begin{tikzcd} \textit{a man is just} \arrow[r, dash] \arrow[d] \arrow[dr, dash, dashed] & \textit{a man is not just} \\ \textit{a man is not unjust} \arrow[r, dash] \arrow[ur, dash, dotted] & \textit{a man is unjust} \arrow[u] \end{tikzcd}$$

Figure 3: Square of opposition for privative propositions.

differently: how are the two squares of opposition in Figures 2 and 3 related to each other? In the ensuing ancient commentary tradition, we find at least two answers to this question. On the one hand, we have the thesis of the *union* of the two squares, proposed by Ammonius in his commentary on *De Interpretatione* (Busse 1897). On the other hand, we have the thesis of the *collapse* between the squares of privative and infinite propositions, proposed specifically by Porphyry, according to the report of Boethius (Meiser 1880).

2.3. *Union of Squares: Proclus and Ammonius*

Ammonius' interpretation emerges as one of the most original of the ancient commentators. As can be seen at the beginning of his commentary on Aristotle's *De Interpretatione*, Ammonius acknowledges that he is reproducing Proclus' teaching,[9] and therefore we follow Correia (2006) in attributing this position to both authors.

[9] "Now, we have recorded the interpretations of our divine teacher Proclus, successor to the chair of Plato and a man who attained the limits of human capacity both in the ability to interpret the opinions of the ancients and in the scientific judgment of the nature of reality. If, having done that, we too are able to add anything to the clarification of the book, we owe great thanks to the god of eloquence" (Blank 2013, p. 1, ll. 8 – 13). Proclus' scholarly output was quite remarkable, but unfortunately most of his works were lost, with the exception of five commentaries on Plato, one on Euclid, and some manuals and monographs. This loss implies that Proclus' ideas on the subject we address here are only accessible second-hand, through Ammonius. Hence, whenever we refer to Ammonius in the remainder of this paper, it should be understood that we are referring to both Proclus and Ammonius, assuming the claim under consideration to be attributable to both. The diagram in Figure 4, however, will be called 'Ammonius' hexagon', since this hexagon is found specifically in Ammonius' text.

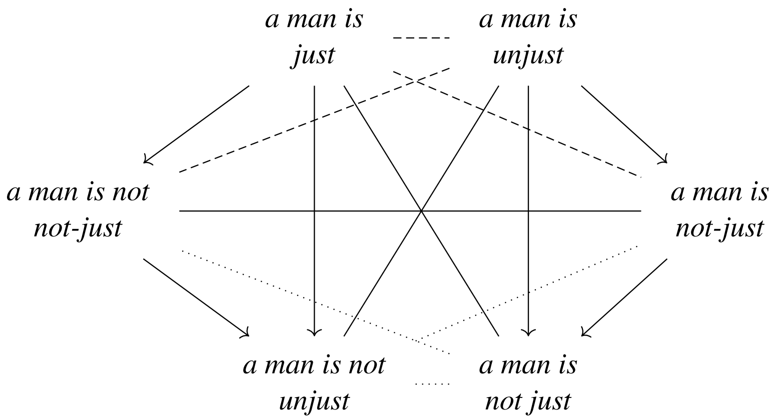


Figure 4: Ammonius' hexagon.

Beyond attempting to solve the problem of interpreting the passage from Aristotle, this proposal independently constitutes a logical theory of privation as an internal negation distinct from infinite negation. In this subsection, we will discuss the rules Ammonius proposes, which will allow us to formulate his hexagon of opposition for singular propositions.[10]

Ammonius' proposal relies on two underlying principles, from which his rules of subalternation follow. First, predicate terms in propositions can be compared to each other, establishing a kind of containment relation. Second, propositional (i.e., copula) negation affects this relation by reversing the containment order. Specifically, Ammonius' proposal for interpreting the passage is that the propositions in question maintain the order shown in Figure 4. This diagram is based on the following rules:

1. "The transposed negative proposition should be greater than the simple affirmative proposition."[11] (Busse 1897, p. 162, ll. 3 – 4)

2. "Then the transposed affirmative proposition will be smaller than the simple negative proposition."[12] (Busse 1897, p. 162, ll. 4 – 5)

3. "Certainly, the privative negative proposition [. . .] is greater than the simple affirmative proposition."[13] (Busse 1897, p. 163, ll. 32 – 34)

[10] Ammonius does not limit himself to these propositions, for he extends his analysis to quantified categorical propositions; cf. Busse (1897, p. 171, l. 21 – p. 172, l. 5) and Correia (1997, pp. 272 – 273). Such propositions are left for further analysis, since they involve the formulation of a more complex diagram that will be explored in a future paper.

[11] ὥστε ἐπὶ πλέον εἴη ἂν ἡ ἐκ μεταθέσεως ἀπόφασις τῆς ἁπλῆς καταφάσεως.

[12] τῆς ἄρα ἀποφάσεως τῆς ἁπλῆς ἐπ' ἔλαττον ἔσται ἡ ἐκ μεταθέσεως κατάφασις.

[13] Next to the privative negative and simple affirmative propositions, the complete text also mentions the negative transposed proposition. Curiously, Ammonius uses the term 'indefinite' (ἀόριστον),

4. "The privative affirmative proposition is smaller than the simple negative proposition."[14] (Busse 1897, p. 164, ll. 1 – 2)

5. "Manifestly, the privative affirmative proposition is smaller than the transposed one."[15] (Busse 1897, p. 164, ll. 18 – 19)

6. "So as, in accordance with truth, the privative negative proposition is greater than the transposed negative proposition."[16] (Busse 1897, p. 164, ll. 22 – 26)

Following Theophrastus,[17] Ammonius uses the term 'transposed' to refer to infinite negation (but recall Footnote 13). This terminological choice is due to the fact that the subalternation on the right side of the square for simple and infinite propositions in Figure 2 goes in the reverse direction of the subalternation on the left side. Hence, a 'transposition' of the propositions on the right must be performed in order to restore the usual direction of subalternation.

Ammonius furthermore uses 'affirmation' and 'negation' to refer to what we would call an 'affirmative' and a 'negative' proposition, respectively. He does not use the term 'negation' to refer to a logical connective. When he mentions, for example, 'the privative affirmative proposition', we will understand that he means a proposition without propositional negation and with privative negation in the predicate, such as 'a man is unjust'. Analogous remarks apply to the other cases as well. This might be confusing from a contemporary linguistic perspective, because a proposition such as 'a man is unjust' could already be considered to have a negative meaning, merely because of the presence of the privative negation in the predicate. Similarly, a proposition like 'a man is not unjust' could be considered affirmative in meaning, because the propositional negation ('not') and the privative negation inside the predicate ('un-') supposedly cancel each other out, yielding a proposition that is either entirely equivalent to 'a man is just' (cf. the law of double negation elimination in classical propositional logic), or at least affirmative in nature (cf. the phenomenon of litotes) (Horn 1989; 2017). This contemporary perspective, which is clearly semantically driven, is equally consistent as Ammonius' more syntactically inspired approach; however, as we wish to formulate Ammonius' rules as faithfully as possible, we have retained his own terminology.

Each of these six rules describes one of the subalternations of the diagram in Figure 4. Furthermore, the contradictions on the diagonals are presupposed.[18] All

rather than the term 'transposed' (ἐκ μεταθέσεως), as in the other rules: "Certainly, the privative negative proposition is greater than the indefinite negative proposition and it happens that it is greater than the simple affirmative proposition". (ἐπὶ πλειόνων γὰρ ἀληθὴς ἡ στερητικὴ ἀπόφασις, ἐφ' ὧν καὶ τὴν ἀόριστον ἀπόφασιν κατελαμβάνομεν ἐπὶ πλέον οὖσαν τῆς ἁπλῆς καταφάσεως.)

[14] ἡ κατάφασις ἡ στερητικὴ ἐπ' ἔλαττον ἔσται τῆς ἁπλῆς ἀποφάσεως.

[15] φανερὸν δὴ ὅτι ἐπ' ἔλαττον ἡ κατάφασις ἡ στερητικὴ τῆς ἐκ μεταθέσεως.

[16] ὥστε ἡ ἀπόφασις ἡ στερητικὴ ἐπὶ πλέον τῆς ἐκ μεταθέσεως ἀποφάσεως ἀληθεύσει.

[17] Cf. Correia (1997, pp. 196 – 202), Mueller (2013, p. 397, ll. 2 – 4), Ierodiakonou (2020) and Oesterle (1962, Book II, 2, 7).

[18] An equivalent formulation is proposed by Correia in (1997, p. 251), (2006), and (2017, p. 9).

remaining relations can be inferred from the contradictions and the subalternations. Ammonius outlines some of these relations:

> But, upon, indeed the unquantified propositions according to contingent matter are mutually true together and the two negations possess the same property.[19] (Busse 1897, p. 172, ll. 22 – 24)

This is not entirely correct, because no two affirmative propositions occurring in Figure 4 (simple, privative, transposed) can be true together, although the three negative propositions can indeed be true together.

Ammonius offers an argument for each rule he proposes. We will analyze his arguments for the first two rules and for the last two rules (the latter are the most relevant ones, as they concern the relation between privative and infinite propositions).[20] Ammonius' justification of the first two rules is summarized in the following passage:

> For suppose that all things exist are one thousand in number, and if the simple statement is true in relation to four hundred, then the negation by transposition will be true in relation to more, say six hundred; therefore, it is evident that among the remaining propositions, the negation will be true in relation to six hundred, while the statement by transposition will be true in relation to four hundred.[21] (Busse 1897, p. 162, ll. 9 – 16)

This argument simultaneously justifies the first two rules. A few lines above, Ammonius has expressed how the predication differs in the case of simple affirmation and transposed negation, using the example of a dog (Busse 1897, p. 161, l. 36) as the subject of the expressions 'he is a just man' and 'he is not an unjust man'. The overall idea is to partition the domain of possible subjects for the various expressions, and to show that some expressions apply to a larger number of entities, while others are more limited.

More concretely, suppose we have three subjects: Socrates (who is a human being and is the most just man), the dog Argos (who is neither a human being nor just), and the judge Anytus (who is a human being and is not just). Ammonius supposes that the totality of entities consists of one thousand objects, and describes

[19] ἀλλ' ἐπὶ μὲν τῶν ἀπροςδιορίστων κατὰ τὴν ἐνδεχομένην ὕλην τάς τε καταφάσεις συναληθεύειν ἀλλήλαις συμβαίνει καὶ τὰς ἀποφάσεις ἅτε ταῖς μερικαῖς ἰσοδυναμούσας.

[20] For a more detailed exposition, see Correia (1997, pp. 249 – 261).

[21] ὑποκείσθωσαν γὰρ τὰ ὄντα πάντα, φέρε εἰπεῖν χίλια τὸν ἀριθμόν, καὶ τῆς καταφάσεως τῆς ἁπλῆς ἀληθευούσης καθ' ὑπόθεσιν ἐπὶ τετρακοσίων ἡ ἐκ μεταθέσεως ἀπόφασις ἐπὶ πλέον ἀληθὴς οὖσα λόγου χάριν ἀληθευέτω ἐπὶ ἑξακοσίων· φανερὸν ἄρα ὅτι τῶν λοιπῶν δύο προτάσεων ἡ μὲν ἁπλῆ ἀπόφασις ἐπὶ τῶν λοιπῶν ἑξακοσίων ἀληθεύσει, ἡ δέ γε ἐκ μεταθέσεως κατάφασις ἐπὶ τῶν λοιπῶν τετρακοσίων παρὰ τὴν οἰκείαν ἀπόφασιν ἀληθεύουσα ἐπ' ἔλαττον ἔσται τῆς ἁπλῆς ἀποφάσεως.

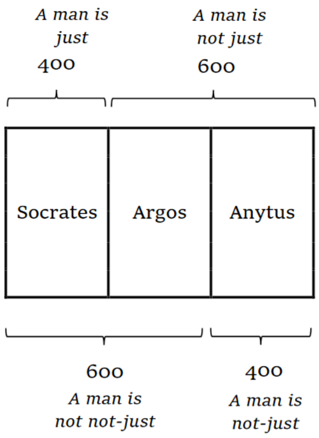


Figure 5: Ammonius' diagram for rules 1 and 2

subsets of entities that make each expression true. The first set consists of four hundred entities, which make true the expression 'is a just man'. By definition, the expression 'is not a just man' will be true of the remaining six hundred entities. A second set consists of four hundred other entities, which make true the expression 'is a not-just man'. Again by definition, the expression 'is not a not-just man' will be true of the remaining six hundred entities. In summary, we thus obtain a partition of our thousand entities into three sets: (i) a set of four hundred just men, who make true 'is a just man' and 'is not a not-just man'; (ii) a set of four hundred men who are not just, and who make true 'is a not-just man' and 'is not a just man'; (iii) a set of two hundred entities who are not men, and who make true 'is not a just man' and 'is not a not-just man'. Figure 5 shows this partition into three cells. The concrete subjects Socrates, Argos and Anytus each belong to one of the cells, as indicated in the figure.

The simple affirmation is 'less than' the transposed negation, because there are two hundred entities (incl. Argos) to which the negation of the infinite predicate applies but to which the affirmation of the simple predicate does not apply. For example, it is true to say 'Argos is not a not-just man', because he is not a man, but it is false to say that 'Argos is a just man'. The dog Argos thus makes the infinite negation true, but the simple affirmation false. The set of entities that make the infinite negation true is strictly larger than the set of entities that makes the simple affirmation true. (The former would include both Socrates and Argos, but the latter only Socrates.)

An analogous reasoning is established for rule 2. The simple negation applies to four hundred entities like the Anytus, who are men who are not just, and furthermore, it also applies to two hundred entities like the dog Argos, who are not men. By contrast, the infinite affirmation only applies to the four hundred entities like Anytus, but excludes the two hundred entities like the dog Argos. After all, the simple negation ('a man is not just') cancels the predicate absolutely, whereas the infinite predicate (‘a man is not-just’) cancels the predicate in a restricted way, requiring that the subject continues to fulfill certain conditions that the simple negation does not impose, e.g., that it exists, that it is a human being, etc.

Let us now turn to Ammonius’ justification for rules 5 and 6, which concern the relation between privative and infinite propositions. For the affirmative propositions (i.e., rule 5), Ammonius argues as follows:

> Indeed, for anyone who is an unjust man, is also a not-just man. Certainly, of a boy, he is not-just, for instance, he is not also unjust, because in the former case ⟨i.e., the boy is not-just⟩, he is in the process of possessing justice, and in the latter case ⟨i.e., the boy is not unjust⟩, neither of the two following cases ⟨holds: the boy⟩ is not disposed by nature to be just, and does not partake in justice.[22] (Busse 1897, p. 164, ll. 19 – 22)

Thomas Aquinas explains that this kind of argument is based on a specific way of relating predicates with the objects that the proposition is truly said of. He states (Oesterle 1962, Book II, 2, 9):

> To make Ammonius’ explanation clear, it must be noted that, as Aristotle himself says, the enunciation, by some power, is related to that of which the whole of what is signified in the enunciation can be truly predicated. The enunciation, ‘Man is just’, for example, is related to all those of which in any way ‘is a just man’ can be truly said. So, too, the enunciation ‘Man is not just’ is related to all those of which in any way ‘is not a just man’ can be truly said.

Using Aquinas’ clarification, Ammonius’ argument can be structured as follows:

1. ‘unjust’ is smaller than ‘not-just’ iff
    - every entity that is unjust, is also not-just,[23] and
    - there is at least one entity that is not-just but not unjust,
2. consider a child:

[22] εἴ τις μὲν γάρ ἐστιν ἄδικος ἄνθρωπος, οὗτος δηλονότι καὶ οὐ δίκαιος, οὐ μέντοι εἴ τις οὐ δίκαιος, ἤδη καὶ ἄδικος, διά τε τὴν μέσην ἕξιν καὶ τὴν μηδετέρου πεφυκυῖάν πω μετέχειν, ὡς ἐπὶ τῶν παίδων. Note that the definition of privation assumed in this passage is similar to that in *Categories* X (Ackrill 1975), *Metaphysics* V (Kirwan 1993), and Ammonius’ commentary on the *Categories* (Cohen & Matthews 2014, p. 95, ll. 16 – 17): “Rather I think it is clear that a privation is spoken of only with respect to a pre-existing possession”. Finally, note that in our translation in the main text, we use ⟨. . . ⟩ to indicate some words that are not literally present in the translated passage, but that we have added in order to increase its comprehensibility.

[23] Cf. the opening sentence of Ammonius’ argument.

- ‘not-just’ can be truly said of the child, but
- ‘unjust’ cannot be truly said of the child.

Since every entity that is unjust, is also not-just, ‘a man is unjust’ entails ‘a man is not-just’. Since there exists at least one entity (viz., a child[24]) that is not-just but not unjust, the reverse entailment does not hold, i.e., ‘a man is not-just’ does not entail ‘a man is unjust’. Putting everything together, ‘a man is unjust’ is smaller than ‘a man is not-just’, or in other words, there is a subalternation from the former to the latter.

Moving from affirmative to negative propositions (i.e., from rule 5 to rule 6), Ammonius’ argument combines the previous strategy and the aforementioned transposition operation:

> Certainly, the privative negative proposition is true applied to a child, since he is not an unjust man, and the transposed negative proposition is false applied to him, since the transposed affirmative proposition is true about a child, i.e. he is not a not-just man.[25] (Busse 1897, p. 164, ll. 23 – 26)

Applying Aquinas’ explanation yields the following structure:

1. ‘not not-just’ is smaller than ‘not unjust’ iff
    - every entity that is not not-just, is also not unjust, and
    - there is at least one entity that is not unjust but that is not-just,
2. consider a child:
    - ‘not unjust’ can be truly said of the child,
      because ‘unjust’ cannot be truly said of the child,
    - ‘not not-just’ cannot be truly said of the child,
      because ‘not-just’ can be truly said of the child.

Once again, Ammonius starts from the idea that privative predicates are smaller than infinite predicates, but because we are dealing with negative propositions now, this order is reversed. We thus find that ‘a man is not not-just’ is smaller than ‘a man is not unjust’, or in other words, there is a subalternation from the former to the latter.

[24]Interestingly, the term ‘παίδων’, which Ammonius uses in this part, can mean ‘child’ as well as ‘slave’, and both translations make sense in this context: neither a child nor a slave have a pre-existing possession of justice, and thus neither of them can be called ‘unjust’.

[25]καὶ γὰρ ἐπὶ τῶν προειρημένων ἕξεων, τῆς τε μέσης καὶ τῆς τῶν παίδων, ἡ μὲν στερητικὴ ἀπόφασις ἀληθής, ἐπεὶ μή εἰσιν ἄνθρωποι ἄδικοι, ἡ μέντοι ἐκ μεταθέσεως ψευδής, ἐπεὶ ἡ κατάφασις αὐτῆς ἀληθής· εἰσὶ γὰρ οὐ δίκαιοι ἄνθρωποι.

These are the two rules that differentiate Ammonius most clearly from the commentator to which we turn next, viz., Porphyry. As we have seen, Ammonius combines the infinite and privative squares by taking their *union*, thus obtaining a hexagon that contains the subalternations we have explained here. Porphyry, on the other hand, offers a totally different yet equally interesting answer, which implies the *collapse* of the two squares.

2.4. *Collapse of Squares: Porphyry's Position*

In his commentary on Aristotle's *De Interpretatione* (Meiser 1880), Boethius presents the following rules, which, according to him, Porphyry maintains regarding the relation between privative and infinite negations.

1. "A simple affirmation is followed by a privative negation, for if it is true to say that there is a just man, it is true to say that there is no unjust man, for he who is just is not unjust."[26] (Meiser 1880, p. 279, ll. 2 – 6)

2. "For, on the other side, a privative affirmation is indeed followed by a simple negation, but a simple negation is not followed by a privative affirmation."[27] (Meiser 1880, p. 280, ll. 4 – 7)

3. "Privative affirmation, which says 'a man is unjust', is equivalent to infinite affirmation, which says 'a man is not-just'."[28] (Meiser 1880, p. 281, ll. 1 – 3)

4. "Privative negation, which is 'a man is not unjust', is equivalent and agrees with the negation which is infinite, 'a man is not not-just'."[29] (Meiser 1880, p. 281, ll. 8 – 11)

5. "Privative negation, which says 'a man is not unjust', follows from simple affirmation, which says 'a man is just', therefore, the same simple affirmation is followed by infinite negation, that is, that which says that 'a man is just' is followed by that which says that 'a man is not not-just'."[30] (Meiser 1880, p. 281, ll. 12 – 14)

6. "Since the privative affirmation, which says that 'man is unjust', was followed by the simple negative, which proposes that 'man is not just', the infinite

[26]simplicem adfirmationem privatoria negatio sequitur. nam si verum est dicere quoniam est iustus homo, verum est dicere quoniam non est iniustus homo. nam qui iustus est non est iniustus.

[27]in diversa enim parte adfirmationem quidem privatoriam sequitur negatio simplex, negationem vero simplicem adfirmatio privatoria non sequitur.

[28]adfirmatio enim privatoria quae dicit est iniustus homo consentit infinitae adfirmationi quae dicit est non iustus homo.

[29]negatio privatoria quae est non est iniustus homo consentit atque concordat ei negationi quae est infinita non est non iustus homo.

[30]sequitur autem simplicem adfirmationem eam quae dicit est iustus homo privatoria negatio quae dicit non est iniustus homo; sequitur igitur eandem ipsam simplicem adfirmationem infinita negatio, id est eam quae dicit est iustus homo ea quae proponit non est non iustus homo.

$$
\begin{tikzcd}
\textit{a man is just} \arrow[r, dash] \arrow[d] & \textit{a man is not just} \\
\textit{a man is not unjust} \arrow[r, dash] \arrow[d, phantom, "|||"] & \textit{a man is unjust} \arrow[u] \arrow[d, phantom, "|||"] \\
\textit{a man is not not-just} \arrow[r, dash] & \textit{a man is not-just}
\end{tikzcd}
$$

Figure 6: Porphyry's square (in Boethius' exposition).

> affirmation, which says that 'man is not-just', is also followed by the simple negation, which says that 'man is not just'."[31] (Meiser 1880, p. 281, l. 24 – p. 282, l. 2)

In accordance with these rules, Boethius presents the diagram shown in Figure 6.[32] This diagram consists of three pairs of contradictory formulas, and the subalternations on the left and right sides go in reverse directions. Based on Boethius' diagram and his exposition of Porphyry's rules, we conclude that Porphyry's position gives rise to the square of opposition shown in Figure 7. In this square, two of the vertices are occupied by two equivalent propositions, viz., the privative and the corresponding infinite proposition. Concerning this equivalence, Porphyry maintains that privative and infinite propositions are merely two distinct syntactic ways of expressing the same underlying meaning. His position on this matter is described by Boethius as follows:

> Let us therefore deal with the infinite and the privative. For privative and infinite propositions, affirmations agree with affirmations, and negations are equivalent with negations in this way; the privative affirmation, which says that 'man is unjust', is equivalent to the affirmation, which says that 'man is not-just', for privative affirmation and infinite affirmation both mean the same thing, and although they differ in some language in their pronunciation, they do not differ in meaning, except only that what the former posits to be unjust, that is, privation, the latter posits to be not-just.[33] (Meiser 1880, p. 280, l. 30 – p. 281, l. 8).

In sum, Porphyry largely finds himself in agreement with Ammonius, except for the relationship between privative and infinite negation: Ammonius holds that there

[31] rursus e diversa parte idem evenit: quoniam adfirmationem privatoriam quae dicit est iniustus homo sequebatur negativa simplex quae proponit non est iustus homo, sequitur quoque infinitam

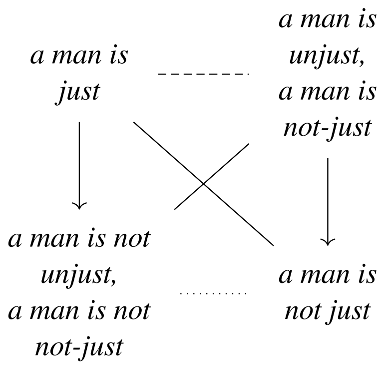


Figure 7: Porphyry's square.

is a subalternation (i.e., one-way entailment) from privative to infinite propositions, while Porphyry takes them to be equivalent to each other. A minor difference concerns terminology (Ammonius talks about 'transposed' propositions, Porphyry about 'infinite' propositions), but this does not affect their respective diagrams. Finally, whereas Ammonius tried to construct an argument for his claim about the (one-way) subalternation from privative to infinite propositions, Porphyry simply states that these two propositions are equivalent to each other, without offering a justification for this claim.

3. Some Logical Background

3.1. *Aristotelian-Alexandrian, Ammonian, Porphyrian and Boolean Systems of Predicate Logic*

We now turn from the historical to the more logical part of the paper. This subsection presents four systems of predicate logic. The first three systems are inspired by the ancient authors presented in Section 2, i.e., Aristotle, Alexander of Aphrodisias, Ammonius and Porphyry, while the fourth system corresponds to contemporary, 'Boolean' predicate logic. The four systems are presented in order of increasing deductive strength.

We begin by defining the language $\mathcal{L}_{ip}$ of infinite and privative negation, which is shared by all the logical systems that we will be dealing with.

adfirmationem quae dicit est non iustus homo simplex negatio quae dicit non est iustus homo.

[32]Cf. Meiser (1880, p. 277, l. 25 – p. 278, l. 3).

[33]his ergo ita positis de infinitis privatoriisque tractemus. privatoriae namque et infinitae adfirmationes adfirmationibus, negationes consentiunt negationibus hoc modo. adfirmatio enim privatoria quae dicit est iniustus homo consentit infinitae adfirmationi quae dicit est non iustus homo. idem enim significant utraeque et privatoria adfirmatio et infinita adfirmatio et quamquam in aliquo sermone prolatione discrepent, tamen significatione nil discrepant, nisi tantum quod quem illa iniustum ponit id est privatoria, haec ponit esse non iustum.

DEFINITION 1 We fix a set of constants $\mathbb{C} = \{a, b, c\ldots\}$, a set of variables $\mathbb{V} = \{x, y, z\ldots\}$ and a set of unary predicate symbols $\mathbb{P} = \{P, Q, R\ldots\}$. The language $\mathcal{L}_{ip}(\mathbb{C}, \mathbb{V}, \mathbb{P})$, which will usually be abbreviated to $\mathcal{L}_{ip}$, is defined by the following BNF:

1. $\Phi \ ::= \ P \mid \overline{P} \mid \widehat{P}$ (where $P \in \mathbb{P}$),
2. $\varphi \ ::= \ \Phi(t) \mid \neg\varphi \mid \varphi \wedge \varphi \mid \forall x\varphi$ (where $t \in \mathbb{C} \cup \mathbb{V}$ and $x \in \mathbb{V}$).

The remaining connectives and quantifiers are defined as usual, by means of $\neg$, $\wedge$ and $\forall$.[34] Note that $\mathcal{L}_{ip}$ is essentially the language of monadic first-order logic, together with the predicate modifiers $\bar{}$ and $\widehat{}$, which correspond to infinite and privative negation, respectively.[35] Also note that these predicate modifiers cannot be applied recursively, so $\overline{P}(a)$ and $\widehat{P}(a)$ are wff's, but $\overline{\overline{P}}(a)$, $\overline{\widehat{P}}(a)$, $\widehat{\overline{P}}(a)$ etc. are not. Technically speaking, it would be fairly straightforward to generalize the language in this direction, but there does not seem to be any (historical-)philosophical motivation for doing so, and hence we will not pursue it.[36]

We now introduce four classes of models on which $\mathcal{L}_{ip}$ can be interpreted. Each

[34]The diagrams that we will study in Section 4 only contain singular statements, like $P(a)$ and $\neg\overline{P}(a)$. Note, however, that the BNF of $\mathcal{L}_{ip}$ also allows for quantified formulas, like $\forall x P(x)$ and $\neg\forall x\widehat{P}(x)$. This anticipates the investigation of more complex diagrams, which we aim to carry out in future work (cf. Section 5).

[35]Sedlár & Šebela (2019) use the same two predicate modifier symbols, albeit with rather different meanings. In particular, our symbol for privative negation, $\widehat{}$, denotes for them a predicate's *range of applicability*, while our symbol for infinite negation, $\bar{}$, denotes what they call *term negation* (which could be the same as our infinite negation). Although Sedlár and Šebela's approach is syntactically quite similar to ours, a crucial difference concerns our treatment of infinite and privative negation as two separate notions. Closely related to this, Sedlár and Šebela's historical discussion is limited to Aristotle, and does not consider any later commentators, as we do here.

[36]Sentences like 'a is not-just' and 'a is unjust' are affirmative, and thus have existential import (i.e., they entail that a exists), whereas 'a is not just' is negative and thus lacks existential import (i.e., it does not entail that a exists). (Also recall Footnote 5.) Based on such considerations, one could also formalize infinite or privative negation in a system of *free logic* (Lambert 2002, Morscher & Hieke 2001), which would also allow us to work exclusively with the Boolean (propositional) negation $\neg$ (i.e., without making use of $\bar{}$ and $\widehat{}$). For example, the sentence 'a is just' would then be formalized as $P(a) \wedge E!(a)$, 'a is not just' as $\neg P(a) \vee \neg E!(a)$, and 'a is not-just' or 'a is unjust' as $\neg P(a) \wedge E!(a)$. Such an approach toward formalizing infinite negation can be found in Bäck (2011) (where the predication with infinite negation is called *metathetic predication*; cf. Footnote 4).

However, we believe that such an approach based on free logic is ultimately inferior to the one presented in this paper, for three reasons. First of all, our choice for the predicate modifiers $\bar{}$ and $\widehat{}$ is explicitly motivated by linguistic and philosophical considerations, since it allows our formalizations to remain more faithful to the historical passages that were analyzed in Section 2 (where 'not-' and 'un-' also modify adjectives, rather than entire sentences). Secondly, free logic is simply not sufficiently expressive for our purposes in this paper. After all, free logic allows us to formalize how simple negations ('a is not just') differ *either* from infinite affirmations ('a is not-just') *or* from privative affirmations ('a is not-just'), but it cannot capture how the latter two differ from each other. Finally, the main advantage of free logic might be that it allows us to elegantly express whether a formula has or lacks existential import. However, we will see later that the logics developed in this paper can do this equally well; cf. Footnote 49.

class of models is a subclass of the ones defined before it. We thus begin with the most comprehensive class of models:

Definition 2 An *Aristotelian-Alexandrian model* is a tuple $\mathbb{M} = \langle D, I \rangle$, where D is a non-empty set (called the domain of $\mathbb{M}$) and I is an interpretation function, i.e., $I(c) \in D$ for all $c \in \mathbb{C}$ and $I(P), I(\overline{P}), I(\widehat{P}) \subseteq D$ for all $P \in \mathbb{P}$. Furthermore, it is required that $I(\overline{P}) \subseteq D \setminus I(P)$ and $I(\widehat{P}) \subseteq D \setminus I(P)$ for all $P \in \mathbb{P}$. The class of all Aristotelian-Alexandrian models is called $\mathcal{M}_{aa}$.

Definition 3 An *Ammonian model* is an Aristotelian-Alexandrian model $\mathbb{M} = \langle D, I \rangle$, which satisfies the additional condition that $I(\widehat{P}) \subseteq I(\overline{P})$ for all $P \in \mathbb{P}$. The class of all Ammonian models is called $\mathcal{M}_{am}$.

Definition 4 A *Porphyrian model* is an Ammonian model $\mathbb{M} = \langle D, I \rangle$, which satisfies the additional condition that $I(\overline{P}) \subseteq I(\widehat{P})$ for all $P \in \mathbb{P}$. The class of all Porphyrian models is called $\mathcal{M}_{po}$.

Definition 5 A *Boolean model* is a Porphyrian model $\mathbb{M} = \langle D, I \rangle$, which satisfies the additional condition that $D \setminus I(P) \subseteq I(\overline{P})$ for all $P \in \mathbb{P}$. The class of all Boolean models is called $\mathcal{M}_{bo}$.

Looking at Definitions 2 – 5, it is immediately clear that $\mathcal{M}_{bo} \subset \mathcal{M}_{po} \subset \mathcal{M}_{am} \subset \mathcal{M}_{aa}$. Note that in any Ammonian model $\mathbb{M} = \langle D, I \rangle$, it holds for all $P \in \mathbb{P}$ that $I(\widehat{P}) \subseteq I(\overline{P})$, whereas based on the historical discussion in Section 2.3, one might rather have expected the condition $I(\widehat{P}) \subset I(\overline{P})$, i.e., requiring that $I(\widehat{P})$ be a *proper* subset of $I(\overline{P})$. The philosophical and technical reasons for imposing the weaker '$\subseteq$'-condition will be explained in detail in Subsection 4.2. Next, note that in any Porphyrian model $\mathbb{M} = \langle D, I \rangle$, it holds for all $P \in \mathbb{P}$ that $I(\widehat{P}) \subseteq I(\overline{P})$ and also $I(\overline{P}) \subseteq I(\widehat{P})$, and hence $I(\widehat{P}) = I(\overline{P})$. Finally, note that in any Boolean model $\mathbb{M} = \langle D, I \rangle$, it holds for all $P \in \mathbb{P}$ that $I(\overline{P}) \subseteq D \setminus I(P)$ and also $D \setminus I(P) \subseteq I(\overline{P})$, and hence $I(\widehat{P}) = I(\overline{P}) = D \setminus I(P)$.[37] This last observation justifies our use of the label 'Boolean' for the models in $\mathcal{M}_{bo}$: even though such models can be used to interpret the predicate modifiers of infinite negation (i.e., $I(\overline{P})$) and privative negation (i.e., $I(\widehat{P})$), they end up identifying both of them with ordinary, Boolean negation (i.e., $D \setminus I(P)$).

We are now in a position to define the semantics of $\mathcal{L}_{ip}$ on these four classes of models. We first introduce some standard auxiliary notions in Definition 6, and then we state the semantic clauses in Definition 7.

Definition 6 Let $\mathbb{M} = \langle D, I \rangle$ be an Aristotelian-Alexandrian model. A *variable assignment on* $\mathbb{M}$ is a function $g\colon \mathbb{V} \to D$. For any $t \in \mathbb{C} \cup \mathbb{V}$, we define

$$[t]^{\mathbb{M},g} := \begin{cases} I(t) & \text{if } t \in \mathbb{C}, \\ g(t) & \text{if } t \in \mathbb{V}. \end{cases}$$

[37]We could equivalently have defined Boolean models using the additional condition $D \setminus I(P) \subseteq I(\widehat{P})$, rather than $D \setminus I(P) \subseteq I(\overline{P})$, since both conditions yield $I(\widehat{P}) = I(\overline{P}) = D \setminus I(P)$.

Finally, given two variable assignments $g, g' \colon \mathbb{V} \to D$ and a variable $x \in \mathbb{V}$, we say that g' is an *x-variant* of g iff $g(y) = g'(y)$ for all variables $y \in \mathbb{V} \setminus \{x\}$.

Definition 7 Let $\mathbb{M} = \langle D, I \rangle$ be an Aristotelian-Alexandrian model, and let g be a variable assignment on $\mathbb{M}$. The semantics of $\mathcal{L}_{ip}$ is defined as follows:

1. $\mathbb{M}, g \models \Phi(t)$ iff $[t]^{\mathbb{M},g} \in I(\Phi)$,
2. $\mathbb{M}, g \models \neg\varphi$ iff $\mathbb{M}, g \not\models \varphi$,
3. $\mathbb{M}, g \models \varphi \wedge \psi$ iff $\mathbb{M}, g \models \varphi$ and $\mathbb{M}, g \models \psi$,
4. $\mathbb{M}, g \models \forall x\varphi$ iff for all x-variants g' of g, it holds that $\mathbb{M}, g' \models \varphi$.

As usual, if $\Gamma \subseteq \mathcal{L}_{ip}$, we write $\mathbb{M}, g \models \Gamma$ to abbreviate that $\mathbb{M}, g \models \gamma$ for all $\gamma \in \Gamma$. Note that Definitions 6 and 7 are 'only' defined for Aristotelian-Alexandrian models. However, since all Boolean, Porphyrian and Ammonian models are themselves Aristotelian-Alexandrian models (recall that $\mathcal{M}_{bo} \subset \mathcal{M}_{po} \subset \mathcal{M}_{am} \subset \mathcal{M}_{aa}$), Definitions 6 and 7 automatically apply to them as well.

We are now ready to define our four logical systems, in the form of four model-theoretic consequence relations $\models_{aa}$, $\models_{am}$, $\models_{po}$ and $\models_{bo}$ (relative to the four classes of models $\mathcal{M}_{aa}$, $\mathcal{M}_{am}$, $\mathcal{M}_{po}$ and $\mathcal{M}_{bo}$, respectively).

Definition 8 Consider $\Gamma \subseteq \mathcal{L}_{ip}$ and $\varphi \in \mathcal{L}_{ip}$. Then:

1. $\Gamma \models_{aa} \varphi$ iff for all Aristotelian-Alexandrian models $\mathbb{M} \in \mathcal{M}_{aa}$ and all variable assignments g on $\mathbb{M}$: if $\mathbb{M}, g \models \Gamma$ then $\mathbb{M}, g \models \varphi$,
2. $\Gamma \models_{am} \varphi$ iff for all Ammonian models $\mathbb{M} \in \mathcal{M}_{am}$ and all variable assignments g on $\mathbb{M}$: if $\mathbb{M}, g \models \Gamma$ then $\mathbb{M}, g \models \varphi$,
3. $\Gamma \models_{po} \varphi$ iff for all Porphyrian models $\mathbb{M} \in \mathcal{M}_{po}$ and all variable assignments g on $\mathbb{M}$: if $\mathbb{M}, g \models \Gamma$ then $\mathbb{M}, g \models \varphi$,
4. $\Gamma \models_{bo} \varphi$ iff for all Boolean models $\mathbb{M} \in \mathcal{M}_{bo}$ and all variable assignments g on $\mathbb{M}$: if $\mathbb{M}, g \models \Gamma$ then $\mathbb{M}, g \models \varphi$.

We finish this subsection by mentioning some important properties of the logics that have been introduced. We first state them formally in Theorem 1, and then provide some informal discussion.

Theorem 1 The following hold:

1. if $\Gamma \models_{aa} \varphi$ then $\Gamma \models_{am} \varphi$,
2. if $\Gamma \models_{am} \varphi$ then $\Gamma \models_{po} \varphi$,
3. if $\Gamma \models_{po} \varphi$ then $\Gamma \models_{bo} \varphi$,

4. $\models_{aa} \forall x(\overline{P}(x) \rightarrow \neg P(x))$,

5. $\models_{aa} \forall x(\widehat{P}(x) \rightarrow \neg P(x))$,

6. $\not\models_{aa} \forall x(\widehat{P}(x) \rightarrow \overline{P}(x))$ but $\models_{am} \forall x(\widehat{P}(x) \rightarrow \overline{P}(x))$,

7. $\not\models_{am} \forall x(\widehat{P}(x) \leftrightarrow \overline{P}(x))$ but $\models_{po} \forall x(\widehat{P}(x) \leftrightarrow \overline{P}(x))$,

8. $\not\models_{po} \forall x(\overline{P}(x) \leftrightarrow \neg P(x))$ but $\models_{bo} \forall x(\overline{P}(x) \leftrightarrow \neg P(x))$,

9. $\not\models_{po} \forall x(\widehat{P}(x) \leftrightarrow \neg P(x))$ but $\models_{bo} \forall x(\widehat{P}(x) \leftrightarrow \neg P(x))$.

PROOF. All items follow directly from Definitions 2 – 7. For example, item 1 follows from the fact that every Ammonian model is, by Definition 3, an Aristotelian-Alexandrian model. Similarly, item 5 follows from the fact that Definition 2 specifies that $I(\widehat{P}) \subseteq D \setminus I(P)$ in all Aristotelian-Alexandrian models $\mathbb{M} = \langle D, I \rangle$. □

Items 1–3 of Theorem 1 state that our four logical systems are linearly ordered by deductive strength, with $\models_{aa}$ being the weakest logic and $\models_{bo}$ being the strongest. Items 4 and 5 state the basic properties of infinite and privative negation, which hold in Aristotelian-Alexandrian logic and thus in all four logical systems. Item 6 describes how Ammonian logic is strictly stronger than Aristotelian-Alexandrian logic; item 7 describes how Porphyrian logic is strictly stronger than Ammonian logic; and items 8 and 9 describe how Boolean logic is strictly stronger than Porphyrian logic. Finally, the second halves of these last two items justify (once more) the label 'Boolean' for $\models_{bo}$: even though we are dealing with a language $\mathcal{L}_{ip}$ that contains the predicate modifiers of infinite negation (i.e., $\overline{P}(x)$) and privative negation (i.e., $\widehat{P}(x)$), they end up being equivalent to ordinary, Boolean negation (i.e., $\neg P(x)$).

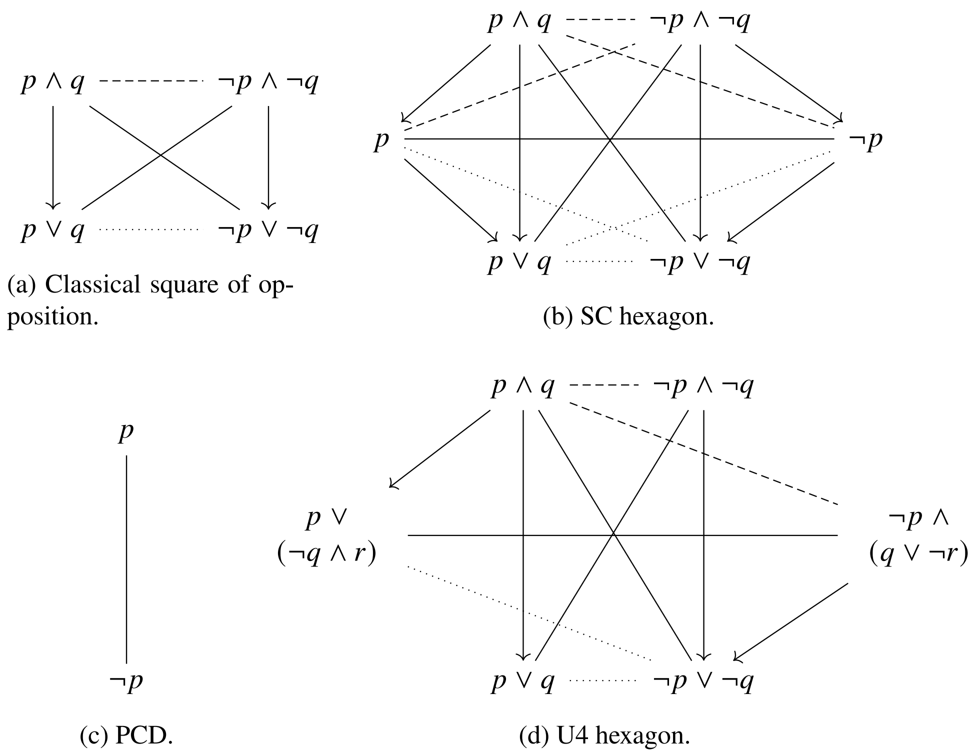
3.2. *Aristotelian Diagrams and Bitstring Semantics*

Aristotelian diagrams, such as the square of opposition, have a rich history in philosophical logic (Johnson 1921, Kraszewski 1956, Kretzmann 1966, Parsons 2017, Read 2012). This subsection introduces some of the formal tools and techniques that are needed to study such diagrams.

DEFINITION 9 Let S be a logical system with Boolean connectives and a model-theoretic semantics $\models_{\mathsf{S}}$. The *Aristotelian relations for* S are defined as follows: two formulas $\varphi, \psi \in \mathcal{L}_{\mathsf{S}}$ are said to be

S-contradictory	iff	$\models_{\mathsf{S}} \neg(\varphi \wedge \psi)$	and	$\models_{\mathsf{S}} \varphi \vee \psi$,
S-contrary	iff	$\models_{\mathsf{S}} \neg(\varphi \wedge \psi)$	and	$\not\models_{\mathsf{S}} \varphi \vee \psi$,
S-subcontrary	iff	$\not\models_{\mathsf{S}} \neg(\varphi \wedge \psi)$	and	$\models_{\mathsf{S}} \varphi \vee \psi$,
in S-subalternation	iff	$\models_{\mathsf{S}} \varphi \rightarrow \psi$	and	$\not\models_{\mathsf{S}} \psi \rightarrow \varphi$.

Furthermore, φ and ψ are said to be



(a) Classical square of opposition.

(b) SC hexagon.

(c) PCD.

(d) U4 hexagon.

Figure 8: Four examples of Aristotelian diagrams in CPL.

$$\begin{array}{llllll} \mathsf{S}\text{-unconnected} & \text{iff} & \not\models_{\mathsf{S}} \neg(\varphi \wedge \psi) & \text{and} & \not\models_{\mathsf{S}} \varphi \vee \psi & \text{and} \\ & & \not\models_{\mathsf{S}} \varphi \rightarrow \psi & \text{and} & \not\models_{\mathsf{S}} \psi \rightarrow \varphi. & \end{array}$$

First of all, note that the four logics introduced in Subsection 3.1 satisfy the requirements of this definition. For example, we say that $P(a)$ and $\widehat{P}(a)$ are *aa*-contrary to each other, because one can easily check that $\models_{aa} \neg(P(a) \wedge \widehat{P}(a))$ and $\not\models_{aa} P(a) \vee \widehat{P}(a)$. Secondly, note that unconnectedness can be viewed as the *absence* of any Aristotelian relation: two (non-equivalent) formulas φ and ψ are unconnected iff they do not stand in a relation of contradiction, contrariety, subcontrariety or subalternation to each other. Finally, note that Definition 9 corresponds exactly with the traditional, more informal approach to the Aristotelian relations (Demey 2019*c*). For example, the clause $\models_{\mathsf{S}} \neg(\varphi \wedge \psi)$ says that there are no S-models $\mathbb{M}$ such that $\mathbb{M} \models \varphi \wedge \psi$, which corresponds to the idea that φ and ψ 'cannot be true together'. Similarly, the clause $\not\models_{\mathsf{S}} \varphi \vee \psi$ corresponds to the idea that φ and ψ 'can be false together'.

Consider a logical system S as in Definition 9, and a finite fragment $\mathcal{F}$ of $\mathcal{L}_{\mathsf{S}}$-formulas. An *Aristotelian diagram for* $(\mathcal{F}, \mathsf{S})$ visualizes all the formulas of $\mathcal{F}$, as well as all the Aristotelian relations holding between them (relative to S). These relations are usually visualized in accordance with the convention that was already described in the caption of Figure 1, i.e., contrariety is visualized by dashed lines, etc.[38] Figure 8 shows four examples of Aristotelian diagrams in classical propositional logic (CPL): (a) a classical square of opposition for $(\mathcal{F}_1, \mathsf{CPL})$, (b) a so-called Sherwood-Czeżowski (SC) hexagon for $(\mathcal{F}_2, \mathsf{CPL})$ (Czeżowski 1955, Kretzmann 1966, Khomskii 2012), (c) a simple pair of contradictories (PCD) for $(\mathcal{F}_3, \mathsf{CPL})$ (Jones 2010, Frijters & Demey 2023) and (d) a so-called unconnectedness-4 (U4) hexagon for $(\mathcal{F}_4, \mathsf{CPL})$ (Kraszewski 1956, Dekker 2015, Demey & Erbas 2024), where $\mathcal{F}_1 := \{p \wedge q, p \vee q, \neg p \wedge \neg q, \neg p \vee \neg q\}$, $\mathcal{F}_2 := \mathcal{F}_1 \cup \{p, \neg p\}$, $\mathcal{F}_3 := \{p, \neg p\}$ and $\mathcal{F}_4 := \mathcal{F}_1 \cup \{p \vee (\neg q \wedge r), \neg p \wedge (q \vee \neg r)\}$ (Frijters & Demey 2023).

In order to compare Aristotelian diagrams with each other, we need the notion of an Aristotelian isomorphism (Demey 2018, Demey & Smessaert 2018*a*):

Definition 10 Consider Aristotelian diagrams for $(\mathcal{F}_1, \mathsf{S}_1)$ and $(\mathcal{F}_2, \mathsf{S}_2)$. An *Aristotelian isomorphism* $f\colon (\mathcal{F}_1, \mathsf{S}_1) \to (\mathcal{F}_2, \mathsf{S}_2)$ is a bijection $f\colon \mathcal{F}_1 \to \mathcal{F}_2$ such that for all $\varphi, \psi \in \mathcal{F}_1$ and for all Aristotelian relations R, it holds that $R_{\mathsf{S}_1}(\varphi, \psi)$ iff $R_{\mathsf{S}_2}(f(\varphi), f(\psi))$.

To illustrate this, consider the classical square of opposition for $\mathcal{F}_5 := \{\Box p, \Diamond p, \Box\neg p, \Diamond\neg p\}$, relative to the normal modal logic KD, as shown in Figure 9. The classical square for $(\mathcal{F}_1, \mathsf{CPL})$ in Figure 8(a) is Aristotelian isomorphic to the classical square for $(\mathcal{F}_5, \mathsf{KD})$; a concrete isomorphism $f\colon \mathcal{F}_1 \to \mathcal{F}_5$ is given by $f(p \wedge q) := \Box p$, $f(p \vee q) := \Diamond p$, $f(\neg p \wedge \neg q) := \Box\neg p$ and $f(\neg p \vee \neg q) := \Diamond\neg p$. Apart from these two squares, no other diagrams shown in Figures 8 and 9 are isomorphic to each other. For example, the classical square for $(\mathcal{F}_1, \mathsf{CPL})$ in Figure 8(a) and the PCD for $(\mathcal{F}_3, \mathsf{CPL})$ in Figure 8(c) are not isomorphic, because there does not even exist a bijection $f\colon \mathcal{F}_1 \to \mathcal{F}_3$. Furthermore, the SC hexagon for $(\mathcal{F}_2, \mathsf{CPL})$ in Figure 8(b) and the U4 hexagon for $(\mathcal{F}_4, \mathsf{CPL})$ in Figure 8(d) are not isomorphic, because the former has *three* pairs of contrary formulas, whereas the latter has only *two* (and it is easy to check that isomorphic diagrams always have the same numbers of Aristotelian relations).

Finally, later in this paper we will make use of *bitstring semantics*. This technique was developed specifically to analyze Aristotelian diagrams (Demey & Smessaert 2018*a*, Smessaert & Demey 2017*b*), and has already found several applications in philosophy and logic (Demey 2018; 2019*a*;*b*). Technically speaking, it can be viewed as a generalization of the truth table semantics and the disjunctive normal form theorem from classical propositional logic, or alternatively, as a logical implementation of the representation theorem for finite Boolean algebras as powerset algebras. Its main advantage is that, instead of providing a semantics for

[38]Often, Aristotelian diagrams are also required to be closed under negation and to exhibit central symmetry, so that the relations of *contradiction* appear on the *diagonals* of the diagram. This is indeed the case for the standard way of showing a classical square of opposition, and also for many other, larger diagrams (Demey & Smessaert 2018*b*). Furthermore, historically speaking, the vast majority of Aristotelian diagrams obey this principle of central symmetry as well. However, there also some notable exceptions (Thiel 1996), including some of the earliest examples of squares of opposition (cf. Figures 1, 2, 3 and 6). All diagrams that appear in the remainder of this paper will indeed satisfy this principle of central symmetry.

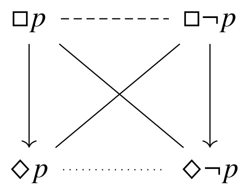


Figure 9: Classical square of opposition in KD.

an entire language $\mathcal{L}_\mathsf{S}$ of a logical system S, it starts from a finite fragment $\mathcal{F} \subseteq \mathcal{L}_\mathsf{S}$ (which typically consists of the formulas that occur in some given diagram), and yields a local, tailor-made semantics $\beta^{\mathcal{F}}_{\mathsf{S}}$ for that specific fragment $\mathcal{F}$.[39] Because of its tailor-made nature, the semantics $\beta^{\mathcal{F}}_{\mathsf{S}}$ provides a concrete combinatorial grip on the logical properties of $\mathcal{F}$; e.g., determining the Aristotelian relations that hold among the formulas of $\mathcal{F}$ becomes a matter of simple bitstring manipulations (examples will be provided below).

Definition 11 Consider a logical system S as in Definition 9 and a finite fragment $\mathcal{F} \subseteq \mathcal{L}_\mathsf{S}$. The *partition induced by $\mathcal{F}$ in* S, denoted $\Pi_\mathsf{S}(\mathcal{F})$, is defined:

$$\Pi_\mathsf{S}(\mathcal{F}) := \{\bigwedge_{\varphi\in\mathcal{F}} \pm\varphi \mid \bigwedge_{\varphi\in\mathcal{F}} \pm\varphi \text{ is } \mathsf{S}\text{-consistent}\},$$

where $+\varphi = \varphi$ and $-\varphi = \neg\varphi$. The elements of $\Pi_\mathsf{S}(\mathcal{F})$ are called *anchor formulas*. The *Boolean closure of $\mathcal{F}$ in* S, denoted $\mathbb{B}_\mathsf{S}(\mathcal{F})$, is defined to be the smallest set $C \subseteq \mathcal{L}_S$ such that (i) $\mathcal{F} \subseteq C$ and (ii) C is closed under the Boolean operations (up to logical equivalence), i.e., for all $\varphi, \psi \in C$, there exist $\alpha, \beta \in C$ such that $\alpha \equiv_\mathsf{S} \varphi \wedge \psi$ and $\beta \equiv_\mathsf{S} \neg\varphi$.

The set $\Pi_\mathsf{S}(\mathcal{F})$ is called a 'partition', because the anchor formulas are (i) jointly exhaustive, that is, $\models_\mathsf{S} \bigvee \Pi_\mathsf{S}(\mathcal{F})$, and (ii) mutually exclusive, that is, $\models_\mathsf{S} \neg(\alpha \wedge \beta)$ for distinct $\alpha, \beta \in \Pi_\mathsf{S}(\mathcal{F})$. It can be shown that every formula in (the Boolean closure of) $\mathcal{F}$ is logically equivalent to a disjunction of anchor formulas: for every $\varphi \in \mathbb{B}_\mathsf{S}(\mathcal{F})$ we have

$$\varphi \equiv_\mathsf{S} \bigvee\{\alpha \in \Pi_\mathsf{S}(\mathcal{F}) \mid \models_\mathsf{S} \alpha \rightarrow \varphi\}.$$

The bitstring semantics $\beta^{\mathcal{F}}_{\mathsf{S}} \colon \mathbb{B}_\mathsf{S}(\mathcal{F}) \rightarrow \{0,1\}^{|\Pi_\mathsf{S}(\mathcal{F})|}$ maps every formula φ in (the Boolean closure of) $\mathcal{F}$ onto its bitstring representation $\beta^{\mathcal{F}}_{\mathsf{S}}(\varphi)$, which is a sequence of $|\Pi_\mathsf{S}(\mathcal{F})|$ bits that keeps track of which anchor formulas occur in this disjunction. Finally, it can be shown that $\beta^{\mathcal{F}}_{\mathsf{S}}$ is a Boolean isomorphism between the Boolean algebras $\mathbb{B}_\mathsf{S}(\mathcal{F})$ and $\{0,1\}^{|\Pi_\mathsf{S}(\mathcal{F})|}$, and therefore also preserves and reflects all the Aristotelian relations.

[39]The *local* nature of bitstring semantics means that $\beta^{\mathcal{F}}_{\mathsf{S}}$ is well-defined for (i.e., can be used to assign semantic intepretations to) the formulas that belong to (the Boolean closure of) $\mathcal{F}$ itself, but *not* for other formulas from $\mathcal{L}_\mathsf{S}$.

For an example, consider the CPL-fragment $\mathcal{F}_1$ from the classical square of opposition in Figure 8(a). An easy calculation yields that $\Pi_{\mathsf{CPL}}(\mathcal{F}_1) = \{\alpha_1 := p \wedge q, \alpha_2 := p \text{ XOR } q, \alpha_3 := \neg p \wedge \neg q\}$. Since $p \vee q \equiv_{\mathsf{CPL}} \alpha_1 \vee \alpha_2$, we represent the formula $p \vee q$ by the bitstring 110; formally: $\beta^{\mathcal{F}_1}_{\mathsf{CPL}}(p \vee q) = 110$. Analogously, the bitstrings for $p \wedge q$, $\neg p \wedge \neg q$ and $\neg p \vee \neg q$ are 100, 001 and 011, respectively. Note, for example, that the CPL-contrariety between the formulas $p \wedge q$ and $\neg p \wedge \neg q$ is reflected in the contrariety between their bitstring representations, 100 and 001 (in the sense that $100 \wedge 001 = 000$ but $100 \vee 001 = 101 \neq 111$, i.e., 100 and 001 'cannot be true together, but can be false together').

4. Aristotelian Diagrams for Privative and Infinite Negation

4.1. *The Aristotelian-Alexandrian Hexagon*

We will now formally analyze and compare the various positions in the discussion on privative and infinite negation, focusing on the kind of Aristotelian diagram that each position gives rise to. Following the order of Section 2, we will start with Aristotle and Alexander of Aphrodisias, then turn to Ammonius, and finally to Porphyry. After these ancient authors, we will also present the contemporary, Boolean perspective. Recall from Section 3 that this order also corresponds to an ordering of the logical systems' deductive strength, with Aristotelian-Alexandrian logic being the weakest system and Boolean logic being the strongest.

The entire discussion revolves around a single fragment $\mathcal{F}_{ip}$ of formulas from $\mathcal{L}_{ip}$. This fragment contains the singular statement $P(a)$, its privative negation $\widehat{P}(a)$, its infinite negation $\overline{P}(a)$, and the (Boolean) negations of these three formulas.[40] Formally, we thus have

$$\mathcal{F}_{ip} := \{P(a), \widehat{P}(a), \overline{P}(a), \neg\overline{P}(a), \neg\widehat{P}(a), \neg P(a)\}.$$

We start by considering this fragment within Aristotelian-Alexandrian logic, $\models_{aa}$. It is an easy exercise to check the following:

1. $\models_{aa} \neg(P(a) \wedge \widehat{P}(a))$	5. $\not\models_{aa} \neg(\widehat{P}(a) \wedge \overline{P}(a))$
2. $\not\models_{aa} P(a) \vee \widehat{P}(a)$	6. $\not\models_{aa} \widehat{P}(a) \vee \overline{P}(a)$
3. $\models_{aa} \neg(P(a) \wedge \overline{P}(a))$	7. $\not\models_{aa} \widehat{P}(a) \rightarrow \overline{P}(a)$
4. $\not\models_{aa} P(a) \vee \overline{P}(a)$	8. $\not\models_{aa} \overline{P}(a) \rightarrow \widehat{P}(a)$

Items 1–2 state that $P(a)$ and $\widehat{P}(a)$ are *aa*-contraries, while items 3–4 state that $P(a)$ and $\overline{P}(a)$ are *aa*-contraries. Furthermore, items 5–8 state that $\widehat{P}(a)$ and $\overline{P}(a)$

[40] The fragment $\mathcal{F}_{ip}$ thus consists exclusively of *singular* statements, since these already suffice to exhibit the intricate interplay between privative, infinite and Boolean negation in the various logical systems under consideration. However, it should be kept in mind that the language $\mathcal{L}_{ip}$ also contains *quantified* statements (recall Footnote 34), so in order to obtain a full understanding of the matter, we will ultimately also need to consider larger fragments, which contain singular as well as quantified statements (each with their privative, infinite and Boolean negations).

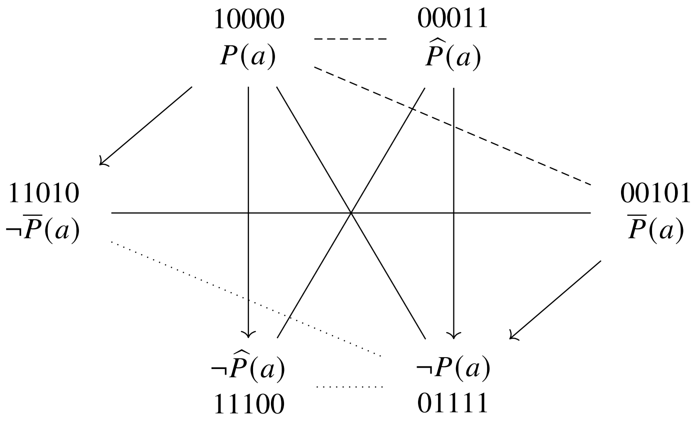


Figure 10: U4 hexagon for $(\mathcal{F}_{ip}, \models_{aa})$.

are *aa*-unconnected. All other Aristotelian relations (or lack thereof) among the formulas of $\mathcal{F}_{ip}$ can be determined in the same way. In summary, the diagram for $(\mathcal{F}_{ip}, \models_{aa})$ is the hexagon shown in Figure 10. This diagram is an unconnectedness-4 (U4) hexagon, just like that for $(\mathcal{F}_4, \mathsf{CPL})$ in Figure 8(d).[41] Although this U4 hexagon does not occur in the ancient sources, it can be viewed as the result of combining the classical square for $\{P(a), \widehat{P}(a), \neg\widehat{P}(a), \neg P(a)\}$ and the classical square for $\{P(a), \overline{P}(a), \neg\overline{P}(a), \neg P(a)\}$, which can explicitly be traced back to Aristotle and Alexander; cf. Figures 2 and 3. Finally, since these latter authors did not comment on the relation between privative and infinite negation, we find that $(\neg)\widehat{P}(a)$ is *aa*-unconnected to $(\neg)\overline{P}(a)$, thus yielding the four relations of unconnectedness in the U4 hexagon in Figure 10.

Finally, in order to give the bitstring semantics for the U4 hexagon for $(\mathcal{F}_{ip}, \models_{aa})$, we first compute the partition that is induced by this hexagon. This partition consists of 5 anchor formulas:

$$\begin{array}{lcllcll}
\Pi_{aa}(\mathcal{F}_{ip}) & = & \{ & \alpha_1 & := & P(a) & \\
 & & & \alpha_2 & := & \neg P(a) \wedge \neg\widehat{P}(a) \wedge \neg\overline{P}(a) & \\
 & & & \alpha_3 & := & \neg\widehat{P}(a) \wedge \overline{P}(a) & \\
 & & & \alpha_4 & := & \widehat{P}(a) \wedge \neg\overline{P}(a) & \\
 & & & \alpha_5 & := & \widehat{P}(a) \wedge \overline{P}(a) & \}.
\end{array}$$

Since $|\Pi_{aa}(\mathcal{F}_{ip})| = 5$, the U4 hexagon for $(\mathcal{F}_{ip}, \models_{aa})$ can be represented by bitstrings of length 5. For example, note that $\overline{P}(a)$ is *aa*-equivalent to $\alpha_3 \vee \alpha_5$, and is thus represented by the bitstring 00101, i.e., $\beta_{aa}^{\mathcal{F}_{ip}}(\overline{P}(a)) = 00101$. All bitstring representations of $(\mathcal{F}_{ip}, \models_{aa})$ can be found in Figure 10.

[41]There thus exists an Aristotelian isomorphism $f\colon (\mathcal{F}_{ip}, \models_{aa}) \to (\mathcal{F}_4, \mathsf{CPL})$.

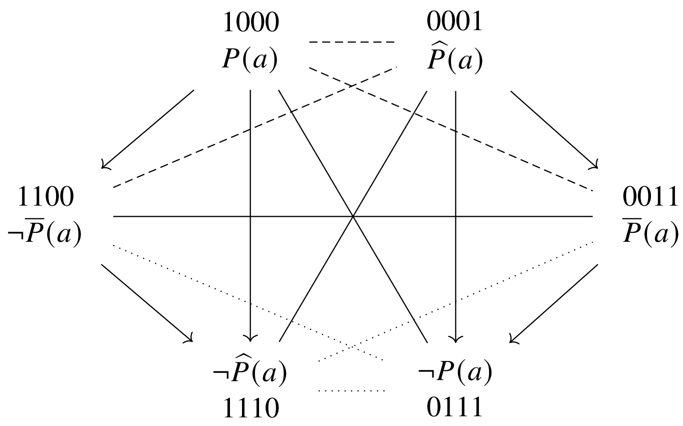


Figure 11: SC hexagon for $(\mathcal{F}_{ip}, \models_{am})$.

4.2. *The Ammonian Hexagon*

We now study the fragment $\mathcal{F}_{ip}$ relative to the system of Ammonian logic, $\models_{am}$. While the formulas $\widehat{P}(a)$ and $\overline{P}(a)$ were *aa*-unconnected, in the stronger Ammonian system they are in *am*-subalternation, as is clear from items 1–2 below. Similarly, the formulas $\widehat{P}(a)$ and $\neg\overline{P}(a)$ go from being *aa*-unconnected to being *am*-contrary, as is clear from items 3–4:

1. $\models_{am} \widehat{P}(a) \rightarrow \overline{P}(a)$
2. $\not\models_{am} \overline{P}(a) \rightarrow \widehat{P}(a)$
3. $\models_{am} \neg(\widehat{P}(a) \wedge \neg\overline{P}(a))$
4. $\not\models_{am} \widehat{P}(a) \vee \neg\overline{P}(a)$

In summary, the diagram for $(\mathcal{F}_{ip}, \models_{am})$ is the hexagon shown in Figure 11. This diagram is a Sherwood-Czeżowski (SC) hexagon, just like that for $(\mathcal{F}_2, \mathsf{CPL})$ in Figure 8(b).[42] This SC hexagon follows exactly the rules laid down by Ammonius, cf. Figure 4. Finally, we compute the partition that is induced by this hexagon:

$$\Pi_{am}(\mathcal{F}_{ip}) = \{ \begin{array}{lll} \alpha'_1 & := & P(a) \\ \alpha'_2 & := & \neg P(a) \wedge \neg\overline{P}(a) \\ \alpha'_3 & := & \neg\widehat{P}(a) \wedge \overline{P}(a) \\ \alpha'_5 & := & \widehat{P}(a) \end{array} \}.$$

Note that, in comparison with $\Pi_{aa}(\mathcal{F}_{ip})$, the anchor formulas α_2 and α_5 simplify to resp. α'_2 and α'_5, and, most importantly, that α_4 goes from being *aa*-consistent to being *am*-inconsistent, and thus drops out of the partition altogether. Consequently, the SC hexagon for $(\mathcal{F}_{ip}, \models_{am})$ can be represented by bitstrings of length 4, which are obtained by systematically deleting the fourth bit position from the corresponding (length-5) bitstrings for $(\mathcal{F}_{ip}, \models_{aa})$. For example, since we already computed that the *aa*-bitstring for $\overline{P}(a)$ is 00101, it follows that the *am*-bitstring for this same formula

[42] There thus exists an Aristotelian isomorphism $f\colon (\mathcal{F}_{ip}, \models_{am}) \rightarrow (\mathcal{F}_2, \mathsf{CPL})$.

is 0011, i.e., $\beta_{am}^{\mathcal{F}_{ip}}(\overline{P}(a)) = 0011$. All bitstring representations of $(\mathcal{F}_{ip}, \models_{am})$ can be found in Figure 11.

Before moving on to the next logical system, we should now explain the philosophical and technical reasons for defining Ammonian models $\mathbb{M} = \langle D, I\rangle$ using the condition $I(\widehat{P}) \subseteq I(\overline{P})$, rather than the stronger $I(\widehat{P}) \subset I(\overline{P})$ (recall Definition 3 and the discussion there).[43] Our goal in introducing the logical system $\models_{am}$ (and thus, the notion of Ammonian model) is to be able to study the SC hexagon for $\mathcal{F}_{ip}$ that it gives rise to, to compare this SC hexagon with the other diagrams for $\mathcal{F}_{ip}$ in other logical systems, etc. (all of which serves the paper's overarching goal of examining and formally reconstructing the ancient discussion on privative and infinite negation). The distinguishing feature is that in $\models_{am}$, we have a subalternation from $\widehat{P}(a)$ to $\overline{P}(a)$ (indeed, we have already seen that in $\models_{aa}$, these formulas are unconnected, and we will soon see that in $\models_{po}$, they are equivalent).[44] To get this *am*-subalternation, we need $\models_{am} \widehat{P}(a) \to \overline{P}(a)$ and $\not\models_{am} \overline{P}(a) \to \widehat{P}(a)$ (cf. items 1 and 2 above). The first condition is guaranteed to hold, by the condition that $I(\widehat{P}) \subseteq I(\overline{P})$ in all Ammonian models. For the second condition, it suffices that there exists at least one Ammonian model in which $I(\overline{P}) \nsubseteq I(\widehat{P})$ (and the existence of such a model is perfectly allowed by Definition 3). Combining this with the first condition, we thus find that there should be *at least one* Ammonian model in which $I(\widehat{P}) \subset I(\overline{P})$.

Based on the historical discussion in Subsection 2.3, one might be inclined to define Ammonian models using this stronger condition $I(\widehat{P}) \subset I(\overline{P})$, so that this will hold in *all* (alternatively defined) Ammonian models. Indeed, this alternative definition will also yield the desired subalternation from $\widehat{P}(a)$ to $\overline{P}(a)$. However, it is important to realize that, from a purely logical perspective, this alternative definition is unnecessarily strong, since as we have seen above, the subalternation from $\widehat{P}(a)$ to $\overline{P}(a)$ only requires that $I(\widehat{P}) \subset I(\overline{P})$ holds in *at least one* (rather than *all*) Ammonian models. Furthermore, with our actual definition of Ammonian model (which has the weak requirement $I(\widehat{P}) \subseteq I(\overline{P})$), we can say that all Porphyrian models are Ammonian models, but not vice versa ($\mathcal{M}_{po} \subset \mathcal{M}_{ap}$), and thus, that $\models_{po}$ is a strictly stronger logical system than $\models_{am}$. If we would choose to work with the alternative definition of Ammonian model (which has the strong requirement $I(\widehat{P}) \subset I(\overline{P})$), we would lose these elegant logical observations (e.g., $\models_{am}$ and $\models_{po}$ would become incomparable in terms of their deductive strength).[45] Finally, from a more historical-philosophical perspective, the case for the alternative definition is not as strong as it may first seem. Indeed, in Subsection 2.3 we have discussed passages by Ammonius which argue that 'unjust' is *strictly* 'smaller than' (i.e., applies to *strictly* fewer entities than) 'not-just', with the strictness deriving from the existence of a child, who is not-just, but not unjust (also recall Footnote 24). However, this concerns one concrete example, which by itself does not allow us to

[43]Thanks to an anonymous reviewer for some useful discussion about this subtle point.

[44]Also cf. items 6 and 7 of Theorem 1.

[45]We will return to these logical observations in Subsection 4.5.

differentiate between the actual and the alternative definitions of Ammonian model, since the difference between these two definitions is ultimately of a quantificational nature: should $I(\widehat{P}) \subset I(\overline{P})$ hold in *at least one*, or rather in *all* Ammonian models? More concretely, recall that even the actual definition still entails that there exists at least one Ammonian model in which $I(\widehat{P}) \subset I(\overline{P})$; we can perfectly view this one Ammonian model as corresponding to the concrete example that was informally described by Ammonius himself.

4.3. *The Porphyrian Square*

Next, we study the fragment $\mathcal{F}_{ip}$ relative to the system of Porphyrian logic, $\models_{po}$. While the formulas $\widehat{P}(a)$ and $\overline{P}(a)$ were in *am*-subalternation, in the stronger Porphyrian system they are *po*-equivalent to each other, as is clear from items 1–2 below. Similarly, the formulas $\widehat{P}(a)$ and $\neg\overline{P}(a)$ go from being *am*-contrary to being *po*-contradictory, as is clear from items 3–4:

1. $\models_{po} \widehat{P}(a) \rightarrow \overline{P}(a)$
2. $\models_{po} \overline{P}(a) \rightarrow \widehat{P}(a)$
3. $\models_{po} \neg(\widehat{P}(a) \wedge \neg\overline{P}(a))$
4. $\models_{po} \widehat{P}(a) \vee \neg\overline{P}(a)$

In summary, the diagram for $(\mathcal{F}_{ip}, \models_{po})$ is the hexagon shown in Figure 12. However, since this hexagon contains two pairs of equivalent formulas, it can be tidied up into a classical square of opposition, as shown in Figure 13.[46] This classical square follows exactly the rules laid down by Porphyry, cf. Figures 6 and 7. Finally, we compute the partition that is induced by this square:

$$\begin{array}{rclrcl} \Pi_{po}(\mathcal{F}_{ip}) & = & \{ & \alpha''_1 & := & P(a) \\ & & & \alpha''_2 & := & \neg P(a) \wedge \neg\widehat{P}(a) \\ & & & \alpha''_5 & := & \widehat{P}(a) \qquad \}. \end{array}$$

Note that, in comparison with $\Pi_{am}(\mathcal{F}_{ip})$, the anchor formula α'_3 goes from being *am*-consistent to being *po*-inconsistent, and thus drops out of the partition altogether. Consequently, the classical square for $(\mathcal{F}_{ip}, \models_{po})$ can be represented by bitstrings of length 3, which are obtained by systematically deleting the third bit position from the corresponding (length-4) bitstrings for $(\mathcal{F}_{ip}, \models_{am})$. For example, since we already computed that the *am*-bitstring for $\overline{P}(a)$ is 0011, it follows that the *po*-bitstring for this same formula is 001, i.e., $\beta^{\mathcal{F}_{ip}}_{po}(\overline{P}(a)) = 001$. Note, in particular, that this same bitstring 001 also gets assigned to $\widehat{P}(a)$, which corresponds to the fact that $\widehat{P}(a)$ and $\overline{P}(a)$ are *po*-equivalent. All bitstring representations of $(\mathcal{F}_{ip}, \models_{po})$ can be found in Figures 12 and 13.

4.4. *The Boolean PCD*

Finally, we study the fragment $\mathcal{F}_{ip}$ relative to the strongest logical system considered in this paper, i.e., the Boolean logic $\models_{bo}$. While the formulas $\overline{P}(a)$ and $\neg P(a)$

[46]The diagram for $(\mathcal{F}_{ip}, \models_{po})$ in Figure 13 is a classical square of opposition, just like that for $(\mathcal{F}_1, \mathsf{CPL})$ in Figure 8(a). Up to logical equivalence, there thus exists an Aristotelian isomorphism $f\colon (\mathcal{F}_{ip}, \models_{po}) \rightarrow (\mathcal{F}_1, \mathsf{CPL})$.

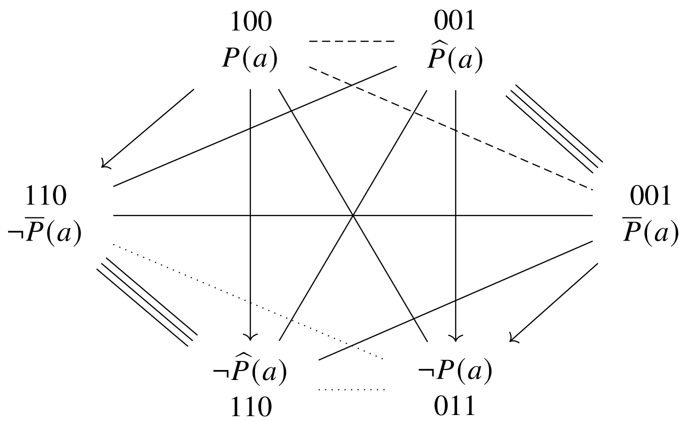


Figure 12: Classical square of opposition for $(\mathcal{F}_{ip}, \models_{po})$, before tidying up.

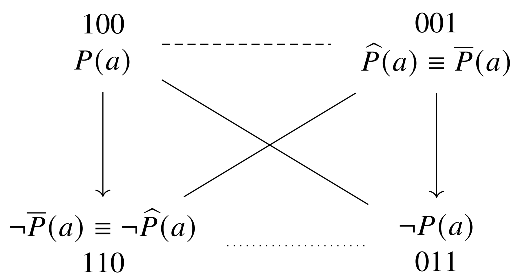


Figure 13: Classical square of opposition for $(\mathcal{F}_{ip}, \models_{po})$, after tidying up.

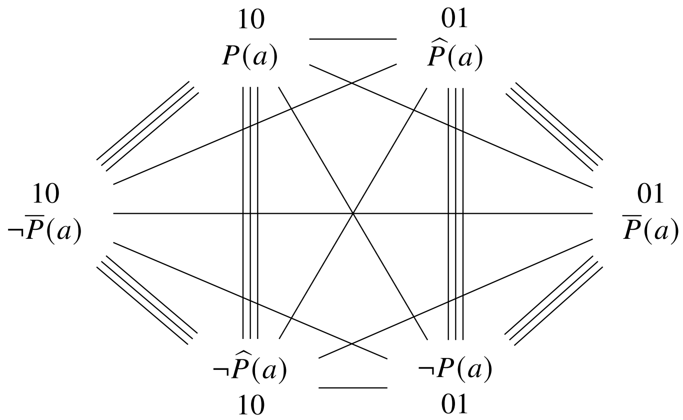


Figure 14: PCD for $(\mathcal{F}_{ip}, \models_{bo})$, before tidying up.

$$\overset{10}{P(a) \equiv \neg\overline{P}(a) \equiv \neg\widehat{P}(a)} \quad \text{———} \quad \overset{01}{\widehat{P}(a) \equiv \overline{P}(a) \equiv \neg P(a)}$$

Figure 15: PCD for $(\mathcal{F}_{ip}, \models_{bo})$, after tidying up.

were in *po*-subalternation, in the stronger Boolean system they are *bo*-equivalent to each other, as is clear from items 1–2 below. Similarly, the formulas $P(a)$ and $\overline{P}(a)$ go from being *bo*-contrary to being *bo*-contradictory, as is clear from items 3–4:

1. $\models_{bo} \overline{P}(a) \to \neg P(a)$
2. $\models_{bo} \neg P(a) \to \overline{P}(a)$
3. $\models_{bo} \neg(P(a) \wedge \overline{P}(a))$
4. $\models_{bo} P(a) \vee \overline{P}(a)$

In summary, the diagram for $(\mathcal{F}_{ip}, \models_{bo})$ is the hexagon shown in Figure 14. However, since this hexagon contains two triples of mutually equivalent formulas, it can be tidied up into a pair of contradictories (PCD), as shown in Figure 15.[47] This PCD clearly shows that in the Boolean system, privative and infinite negation end up being equivalent to Boolean negation. On the right side of the PCD, we find three equivalent ways of negating $P(a)$. On the left side of the PCD, the two negations in $\neg\overline{P}(a)$ and in $\neg\widehat{P}(a)$ cancel each other out, so these two formulas simplify to the equivalent $P(a)$. Finally, we compute the partition that is induced by this PCD:

$$\Pi_{po}(\mathcal{F}_{ip}) = \{ \begin{array}{lll} \alpha_1''' & := & P(a) \\ \alpha_5''' & := & \neg P(a) \end{array} \}.$$

Note that, in comparison with $\Pi_{po}(\mathcal{F}_{ip})$, the anchor formula α_5'' becomes equivalent to a purely Boolean formula, and, most importantly, that the anchor formula

[47] The diagram for $(\mathcal{F}_{ip}, \models_{bo})$ in Figure 15 is a PCD, just like that for $(\mathcal{F}_3, \mathsf{CPL})$ in Figure 8(c). Up to logical equivalence, there thus exists an Aristotelian isomorphism $f\colon (\mathcal{F}_{ip}, \models_{bo}) \to (\mathcal{F}_3, \mathsf{CPL})$.

α_2'' goes from being *po*-consistent to being *bo*-inconsistent, and thus drops out of the partition altogether. Consequently, the PCD for $(\mathcal{F}_{ip}, \models_{bo})$ can be represented by bitstrings of length 2, which are obtained by systematically deleting the second bit position from the corresponding (length-3) bitstrings for $(\mathcal{F}_{ip}, \models_{po})$. For example, since we already computed that the *po*-bitstring for $\overline{P}(a)$ is 001, it follows that the *bo*-bitstring for this same formula is 01, i.e., $\beta_{bo}^{\mathcal{F}_{ip}}(\overline{P}(a)) = 01$. All bitstring representations of $(\mathcal{F}_{ip}, \models_{bo})$ can be found in Figures 14 and 15.

4.5. *Summary*

In this section, we have formalized the ancient discussion on privative and infinite negation, by defining the fragment $\mathcal{F}_{ip}$ and studying it relative to the logics $\models_{aa}$, $\models_{am}$, $\models_{po}$ and $\models_{bo}$. In particular, we have shown that relative to these four different logical systems, this fragment gives rise to four different Aristotelian diagrams:

- The diagram for $(\mathcal{F}_{ip}, \models_{aa})$ is an unconnectedness-4 (U4) hexagon.
- The diagram for $(\mathcal{F}_{ip}, \models_{am})$ is a Sherwood-Czeżowski (SC) hexagon.
- The diagram for $(\mathcal{F}_{ip}, \models_{po})$ is a classical square of opposition.
- The diagram for $(\mathcal{F}_{ip}, \models_{bo})$ is a pair of contradictories (PCD).

These four diagrams are all genuinely different from each other, in the sense that there exists no Aristotelian isomorphism between any two of them.[48] The fact that a single fragment can give rise to different Aristotelian diagrams illustrates the *logic-sensitivity* of these diagrams. In recent years, this phenomenon has been studied extensively from a purely logical perspective (Demey 2015; 2021, Demey & Smessaert 2018*a*, Demey & Frijters 2023), but it is interesting to also observe its appearance in the context of formalizing an ancient philosophical discussion.

Finally, it is worthwhile to have another look at the bitstrings for these four Aristotelian diagrams, and in particular, their underlying partitions:

$\Pi_\bullet(\mathcal{F}_{ip})$	α_1	α_2	α_3	α_4	α_5
aa	$P(a)$	$\neg P(a) \wedge \neg\widehat{P}(a) \wedge \neg\overline{P}(a)$	$\neg\widehat{P}(a) \wedge \overline{P}(a)$	$\widehat{P}(a) \wedge \neg\overline{P}(a)$	$\widehat{P}(a) \wedge \overline{P}(a)$
am	$P(a)$	$\neg P(a) \wedge \neg\overline{P}(a)$	$\neg\widehat{P}(a) \wedge \overline{P}(a)$		$\widehat{P}(a)$
po	$P(a)$	$\neg P(a) \wedge \neg\widehat{P}(a)$			$\widehat{P}(a)$
bo	$P(a)$				$\neg P(a)$

[48] Also note that as we progress through *stronger* logics (from $\models_{am}$ to $\models_{po}$ to $\models_{bo}$), the resulting diagrams become *smaller* (from hexagon to square to PCD), i.e., more and more formulas end up being equivalent to each other. This is in line with the well-known observation that there is an inverse correlation between a logic's *deductive* strength (i.e., the number of formulas it proves to be tautologies) and its *discriminatory* strength (i.e., the number of distinctions between non-equivalent formulas it is able to maintain) (Nelson 1959, Humberstone 2005).

Intuitively, these formulas can be viewed as ordered from the most positive/affirmative (α_1) on the far left to the most negative (α_5) on the far right. In the weakest logic, $\models_{aa}$, the entire spectrum is 'populated', with α_1, α_2, α_3, α_4 and α_5 all representing genuine (i.e., *aa*-consistent) possibilities. However, as we move to stronger and stronger logical systems, more and more anchor formulas become inconsistent and thus drop out of the spectrum. Finally, we arrive in the strongest logic, $\models_{bo}$, where the 'spectrum' has been reduced to its two endpoints, α_1 and α_5.[49]

5. Conclusion

In this paper, we have presented some Aristotelian diagrams that arise in the discussion on infinite and privative negation from late antiquity. First, we presented a synthesis of the discussion, focusing on the positions of Aristotle (and Alexander of Aphrodisias), Ammonius Hermiae (and Proclus), and Porphyry. Next, we formulated logical systems corresponding to each of these positions, as well as a contemporary, Boolean system of logic. Finally, we introduced the six-formula fragment $\mathcal{F}_{ip}$ and studied the Aristotelian diagrams it gives rise to (relative to each of the four logical systems under consideration), as well as their respective bitstring semantics. In particular, we argued that Aristotle and Alexander's position gives rise to an unconnected-4 (U4) hexagon, while Ammonius and Proclus' ideas produce a Sherwood-Czeżowski (SC) hexagon. In Porphyry's position, these hexagons collapse to a classical square of opposition, and in contemporary, Boolean logic, they collapse even further, to a pair of contradictories (PCD). These transitions between diagrams visually exhibit the logical variations that occur when changing the underlying conception of infinite and privative negation.

There are several avenues for further research. From a systematic perspective, it could be interesting to connect the discussion in this paper with ongoing research in philosophy of religion (e.g., 'atheism' as the privative negation of 'theism';

[49] We return one more time to the issue of free logic and existential import. As was already explained in Footnote 36, free logic is insufficiently expressive to deal with infinite and privative negation together, so for the sake of concreteness, let's focus exclusively on infinite negation. In free logic, our formulas $P(a)$ and $\overline{P}(a)$ would be rendered as $P(a) \wedge E!(a)$ and $\neg P(a) \wedge E!(a)$, respectively. Consequently, the anchor formula α_2 would become $\neg(P(a) \wedge E!(a)) \wedge \neg(\overline{P}(a) \wedge E!(a))$, which is equivalent to $\neg E!(a)$, and thus states that a does not exist. This means that we can determine whether a given formula has or lacks existential import, simply by looking at its second bit position (i.e., the bit position corresponding to α_2). For example, relative to the logics $\models_{aa}$, $\models_{am}$ and $\models_{po}$, the formula $\neg P(a)$ is represented by the bitstrings 01111, 0111 and 011, respectively. All these bitstrings have a value 1 in their second bit position, which means that $\neg P(a)$ is *true* in the case described by α_2, i.e., in case a does not exist. This means exactly that $\neg P(a)$ lacks existential import. By contrast, the formula $\overline{P}(a)$ is represented by the bitstrings 00101, 0011 and 001 in the three aforementioned logics. All these bitstrings have a value 0 in their second bit position, which means that $\overline{P}(a)$ is *false* in the case described by α_2, i.e., in case a does not exist. This means exactly that $\overline{P}(a)$ has existential import. Finally, note that if we move to the strongest logic, $\models_{bo}$, the formula α_2 becomes inconsistent (or in other words, $E!(a)$ becomes a tautology), and thus the question of existential import simply does not arise anymore (because in $\models_{bo}$, all singular terms are assumed to exist).

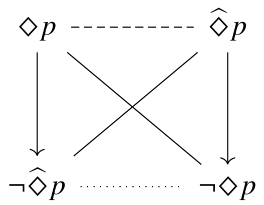


Figure 16: Purported modal square of opposition, in which ‘impossible’ is the privative negation of ‘possible’.

Demey 2019*a*, García-Cruz & Espinoza Ramos 2022), pragmatics (e.g., the status of litotes, such as ‘not unhappy’; Horn 1989; 2017), modal logic (e.g., ‘impossible’ as the privative negation of ‘possible’; García-Cruz 2017, Geudens & Demey 2021) and semantics (e.g., the duality behavior of privative and infinite negation; Demey & Smessaert 2016; 2020, Smessaert & Demey 2017*a*, Löbner 1990). For example, viewing ‘impossible’ as the privative negation of ‘possible’ might lead to the square of opposition shown in Figure 16 (with $\widehat{\ }$ now acting as a modality modifier, rather than a predicate modifier). The subalternation on the left part of this square seems to suggest that ‘not impossible’ ($\neg\widehat{\Diamond} p$) is a strictly weaker version of ‘possible’ ($\Diamond p$), which is in line with the pragmatic (litotes) perspective on these words (Horn 2017, esp. p. 89).

Staying closer to the context of the present paper, it bears emphasizing that our fragment $\mathcal{F}_{ip}$ only contains *singular* statements, since these were also the focus of the historical passages that we analyzed. However, the language $\mathcal{L}_{ip}$ also contains *quantifiers* ($\forall/\exists$), which readily suggests the construction of larger, more complex diagrams (recall Footnotes 34 and 40). Such diagrams can also be traced back to the ancient sources; for example, Ammonius outlined a quantificational fragment, containing six more statements (recall Footnote 10). In future work, we plan to investigate such diagrams for the interaction between quantifiers and infinite/privative negation in more detail.

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