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Tutorial: An Introduction to Logical Geometry: part III

Visual-geometric properties of diagrams

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Bitstrings of length 4

Modal Logic S5	Propositional Logic	bitstrings level 1	bitstrings level 3	Propositional Logic	Modal Logic S5
$\Box p$	$p \wedge q$	1000	0111	$\neg(p \wedge q)$	$\neg\Box p$
$\neg\Box p \wedge p$	$\neg(p \rightarrow q)$	0100	1011	$p \rightarrow q$	$\Box p \vee \neg p$
$\Diamond p \wedge \neg p$	$\neg(p \leftarrow q)$	0010	1101	$p \leftarrow q$	$\neg\Diamond p \vee p$
$\neg\Diamond p$	$\neg(p \vee q)$	0001	1110	$p \vee q$	$\Diamond p$

Modal Logic S5	Propositional Logic	bitstrings level 2/0	bitstrings level 2/4	Propositional Logic	Modal Logic S5
p	p	1100	0011	$\neg p$	$\neg p$
$\Box p \vee (\Diamond p \wedge \neg p)$	q	1010	0101	$\neg q$	$\neg\Diamond p \vee (\neg\Box p \wedge p)$
$\Box p \vee \neg\Diamond p$	$p \leftrightarrow q$	1001	0110	$\neg(p \leftrightarrow q)$	$\neg\Box p \wedge \Diamond p$
$\Box p \wedge \neg\Box p$	$p \wedge \neg p$	0000	1111	$p \vee \neg p$	$\Box p \vee \neg\Box p$

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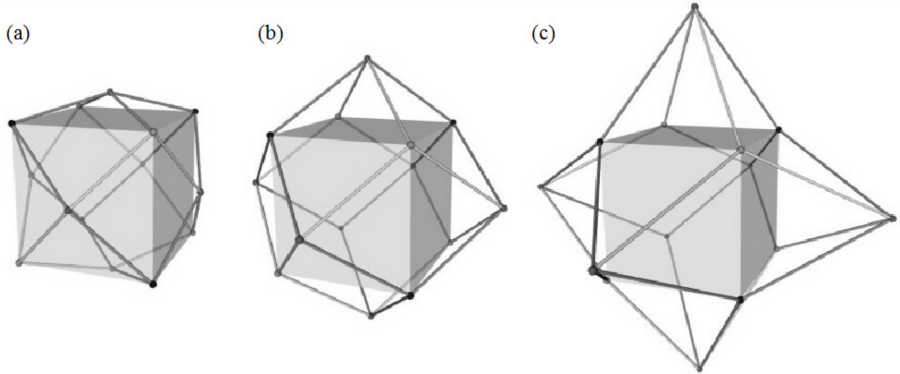
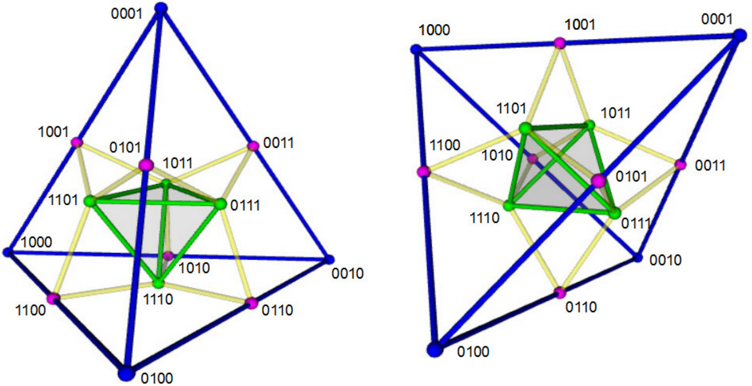
- (a) tetra-hexahedron (Sauriol)
- (b) **rhombic dodecahedron = RDH** (Smessaert-Demey)
- (c) tetra-icosahedron (Moretti-Pellissier)

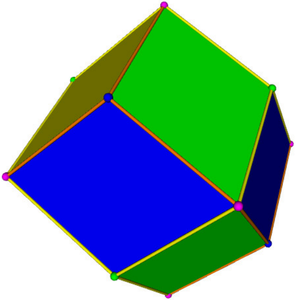


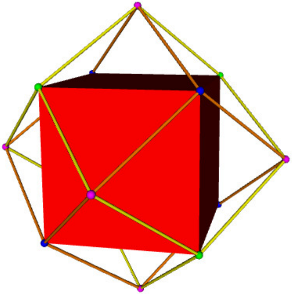
- (d) nested tetrahedron (Lewis, Dubois-Prade)

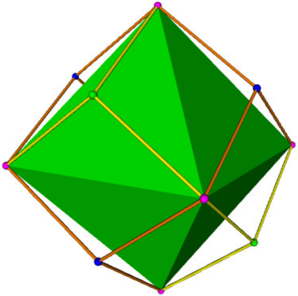


the rhombic dodecahedron (RDH)

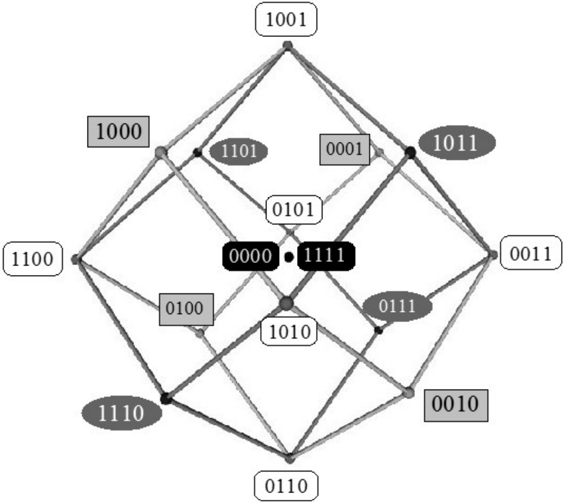
cube	$+$	octahedron	$=$	cuboctahedron	$\overset{dual}{\Longrightarrow}$	**rhombic dodecahedron**
Platonic		Platonic		Archimedean		Catalan
6 faces		8 faces		14 faces		**12 faces**
8 vertices		6 vertices		12 vertices		**14 vertices**
12 edges		12 edges		24 edges		**24 edges**







cube $= 4 \times$ L1 $+ 4 \times$ L3 / **octahedron** $= 6 \times$ L2 / **center** $=$ L0 $+$ L4



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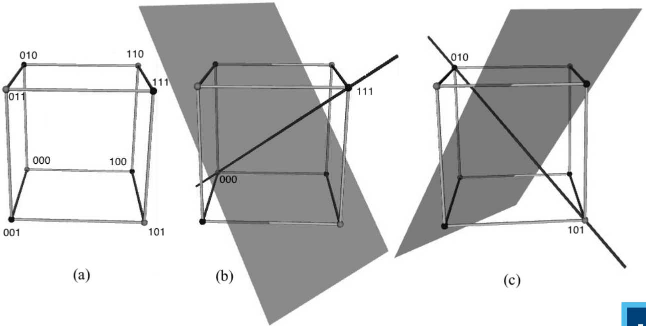
Contradiction relation is visualized using the **central symmetry** of RDH:

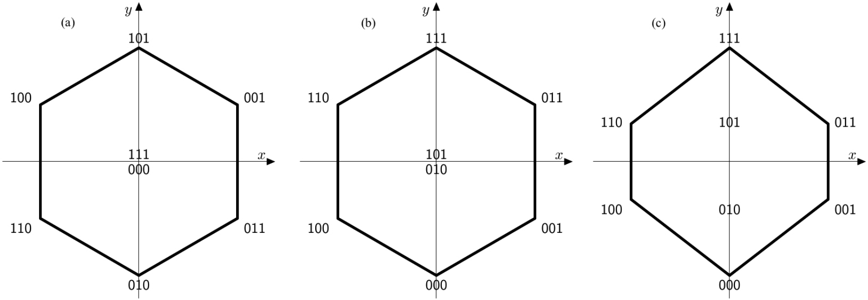
- contradictory bitstrings (e.g. 1100 and 0011) occupy diametrically opposed vertices
- the negation of a bitstring is located at a maximal (Euclidean) distance from that bitstring.
- nearly all Aristotelian diagrams discussed in the literature observe central symmetry ("contradictories are diagonals")

- in (a) the cube consists of 4 pairs of diametrically opposed vertices:
  - 3 contingent pairs: 101—010, 110—001, 011—100
  - 1 non-contingent pair: 111—000
  - each pair defines a projection axis for a **vertex-first projection**:
- in (b) projection along 111—000 axis
- in (c) projection along 101—010 axis



(a) (b) (c)

- the **vertex-first projections** from 3D cube to 2D hexagon:
  - (a) projection along 111—000 ⇒ Aristotelian diagram (JSB)
  - (b) projection along 101—010 ⇒ Hasse diagram (almost)
- if we slightly 'nudge' the projection axis 101—010, we get:
  - (c) projection 'along' 101—010 ⇒ Hasse diagram



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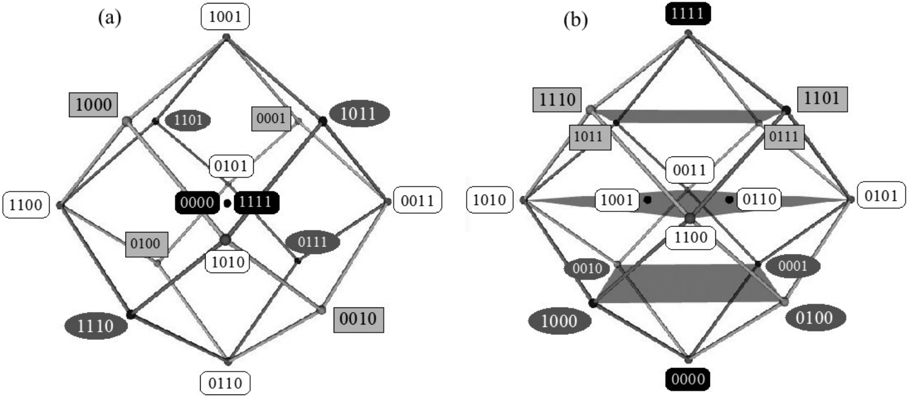
- Aristotelian and Hasse diagram: both **vertex-first projections** of cube
  - Aristotelian diagram: project along the entailment direction
  - Hasse diagram: project along another direction
- the non-contingent formulas $\bot$ and $\top$
  - Hasse diagrams: begin- and endpoint of the $\leq$-ordering
  - Aristotelian diagrams: $\bot$ and $\top$ usually *not* visualized
  - Sauriol, Smessaert, etc.: $\bot$ and $\top$ coincide in the center of symmetry
- the general direction of the entailments
  - Hasse diagrams: all entailments go upwards
  - Aristotelian diagrams: no single shared direction
- visualization of the levels
  - Hasse diagrams: levels L_i are visualized as horizontal hyperplanes
  - Aristotelian diagrams: no uniform visualization of levels

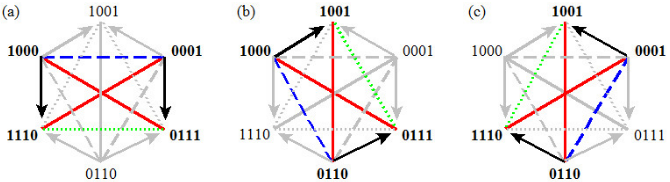
Aristotelian versus Hasse diagrams

the **vertex-first projections** from 4D hypercube to 3D RDH:

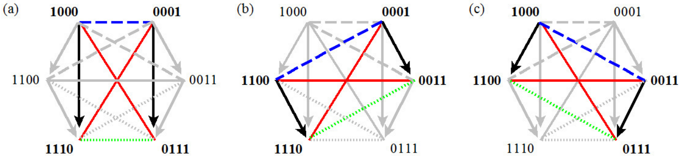
- (a) as an Aristotelian diagram (Moretti, Smessaert, etc.)
- (b) as a Hasse diagram (Zellweger, Kauffman, etc.)



3 squares embedded in (strong) Jacoby-Sesmat-Blanché hexagon (JSB)

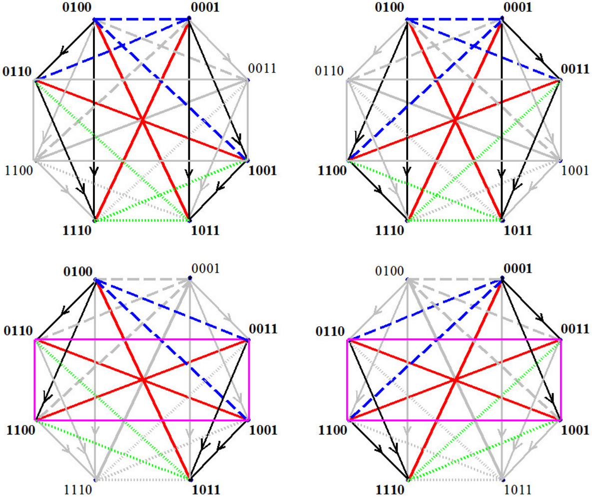


3 squares embedded in Sherwood-Czezowski hexagon (SC)



Aristotelian subdiagrams

4 hexagons embedded in Buridan octagon



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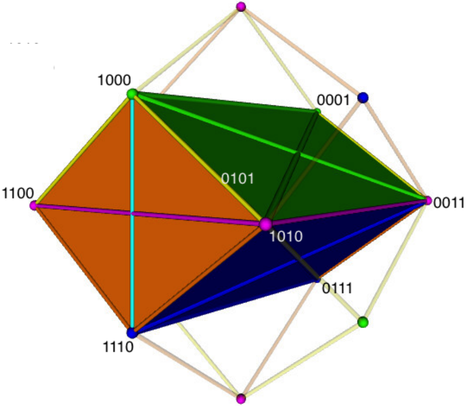
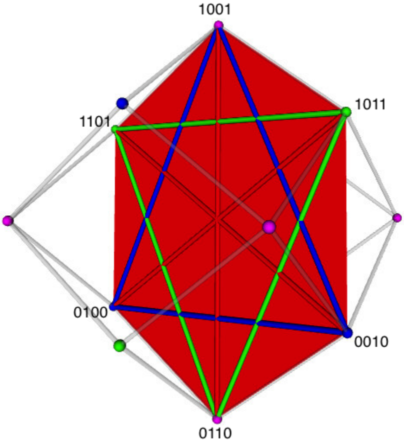
- logical complementarity between Buridan diagram and JSB diagram
  - $\mathbb{B}_4$ has 16 bitstrings (14 after excluding 1111 and 0000)
  - 8 bitstrings have $\neq$ values in bit positions 1 and 4 $\Rightarrow$ Buridan diagram
  - 8 bitstrings have $=$ values in bit positions 1 and 4; 6 after excluding 1111 and 0000 $\Rightarrow$ JSB diagram

1000	**0111**	**1001**	**0110**
0001	**1110**	**1101**	**0010**
1100	**0011**	**1011**	**0100**
0101	**1010**	(**0000**)	(**1111**)

- geometric complementarity between rhombicube and hexagon
  - Buridan embedded inside RDH: rhombicube $\Rightarrow$ partition
  - JSB embedded inside RDH: hexagon of RDH
- rhombicube visualization of Buridan diagram
  - geometric complementarity with JSB hexagon
  - reminder of underlying logical complementarity



1000
0001
0101
1100
0011
1010
0111
1110



1001
1011
1101
0100
0010
0110

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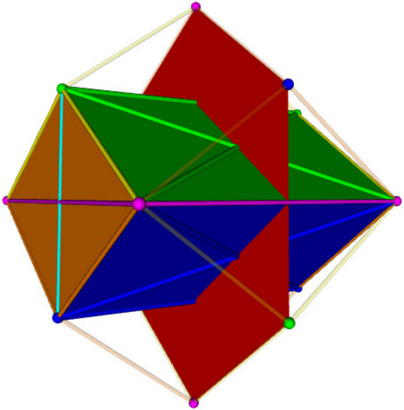

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- Aristotelian families are defined in terms of logical properties
  - Aristotelian relations
    - classical square: 2 CD, 1 C, 1 SC, 2 SA
    - degenerate square: 2 CD
  - Boolean structure
    - classical square: Boolean closure is (isomorphic to) $\mathbb{B}_3$
    - degenerate square: Boolean closure is (isomorphic to) $\mathbb{B}_4$
- diagrams belonging to different Aristotelian families are not *informationally equivalent* (Larkin & Simon)
  - visualize different logical structures
  - differences between diagrams $\leftrightsquigarrow$ differences between logical structures
- if we focus on diagrams belonging to the same Aristotelian family, we notice that different authors still use vastly different diagrams

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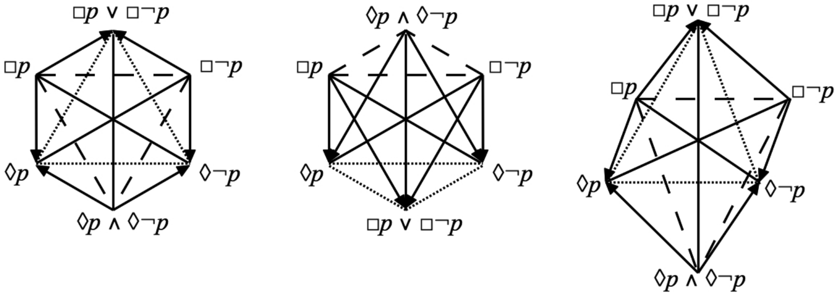
Computational equivalence

- different visualizations of the JSB hexagon:



- these diagrams are *informationally equivalent*, but not *computationally equivalent* (Larkin & Simon)
  - visualize one and the same logical structure
  - visual differences might influence diagrams' effectiveness (user comprehension of the underlying logical structure)

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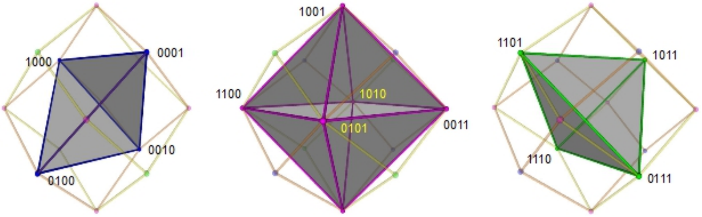
Overview

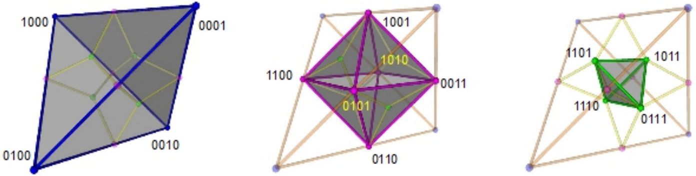
- **Congruity Principle**:
  The content/structure of the visualization should correspond to the content/structure of the desired mental representation.
- in good (cognitively helpful) Aristotelian diagrams, the diagrams' shape helps to visualize the logical properties and relations

[abstract-logical]		[visual-geometric]
properties, relations among sets of formulas	$\longleftarrow$ isomorphism $\longrightarrow$ congruity	shape characteristics of the diagrams

(Corin Gurr, Barbara Tversky)

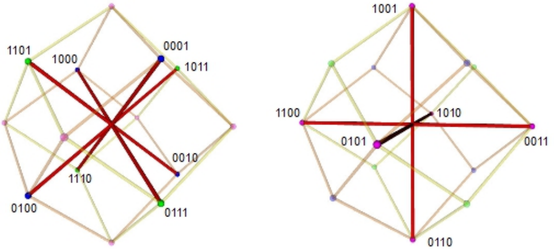


rhombic dodecahedron

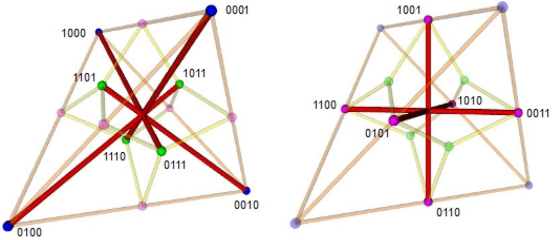


nested tetrahedron

rhombic dodecahedron



nested tetrahedron



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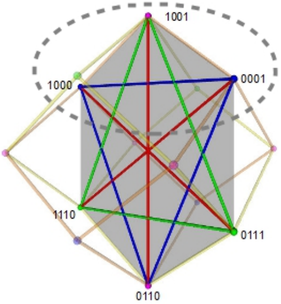
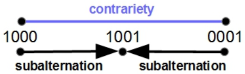
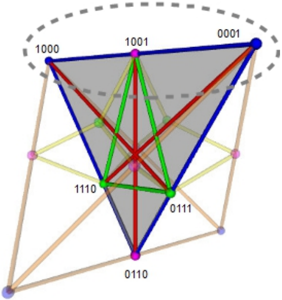
Apprehension Principle:
The content/structure of the visualization should be readily and correctly perceived and understood







rhombic dodecahedron
no collinearity

nested tetrahedron
collinearity

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1. Introduction
2. Geometry: the rhombic dodecahedron (RDH)
3. Geometry: Aristotelian versus Hasse diagrams
4. Geometry: subdiagrams and complementarity
   - Subdiagrams
   - Complementarity
5. Geometry: diagram design principles
   - Informational vs computational equivalence
   - Congruity
   - Apprehension
6. Summary Part III
7. General Conclusion and Prospects

- Part I: Decorations and bitstrings
  - Bitstrings: the basics
  - Decorations: applications in logic, linguistics and cognition
  - Bitstrings anno 2016
- Part II: Abstract-logical properties of diagrams
  - Logic: opposition relations versus implication relations
  - Logic: logic-sensitivity and Boolean subtypes
  - Logic: Aristotelian relations versus duality relations
- Part III: Visual-geometric properties of diagrams
  - Geometry: the rhombic dodecahedron (RDH)
  - Geometry: Aristotelian versus Hasse diagrams
  - Geometry: subdiagrams and complementarity
  - Geometry: diagram design principles

Future work in LG will focus on three main points:

1. Building an **overall typology** with two orthogonal perspectives:
   - all diagrams (independent of number of PCDs) that can be coded with bitstrings of given length:
     - length 3 = internal structure of $\mathbb{B}_3$/JSB = 3 PCDs + 3 squares
     - length 4 = internal structure of $\mathbb{B}_4$/RDH = 7 PCDs + 21 squares + ...
     - length 5 = internal structure of $\mathbb{B}_5$/??? = 15 PCDs + ? squares + ...
   - all diagrams (independent of bitstring length) that contain a given number of PCDs
     - 2 PCDs (square) = classical/degenerate
     - 3 PCDs (hexagon) = JSB/SC/U4/U8/U12
     - 4 PCDs (octagon) = 18 types of octagons (incl. Béziau/Buridan, ...)

2. Building an on-line **corpus/database** of diagrams:
   - service to the research community
   - interface for user submission of contributions

3. Case studies: new **applications/decorations**:
   - contemporary:
     - knowledge representation & AI
     - weak axiomatizations of probability theory
     - proportionality quantfiers => see tomorrow
   - historical:
     - Avicenna
     - Abelard/Buridan
     - Keynes/Johnson

Thank you!

More info: www.logicalgeometry.org