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Tutorial: An Introduction to Logical Geometry: part II

Abstract-logical properties of diagrams

Hans Smessaert & Lorenz Demey

Fifth World Congress on the
Square of Oppositions

Easter Island, November 2016

- Part I: Decorations and bitstrings
  - Bitstrings: the basics
  - Decorations: applications in logic, linguistics and cognition
  - Bitstrings anno 2016
  - Summary Part I: Interdisciplinarity of LG
- Part II: Abstract-logical properties of diagrams
  - Logic: opposition relations versus implication relations
  - Logic: logic-sensitivity and Boolean subtypes
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- Part III: Visual-geometric properties of diagrams
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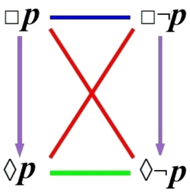
- *Aristotelian square* as the visual representation of a fragment of the *Aristotelian geometry* (diagrams visualize modulo logical equivalence)
- geometry = formulas and relations between them
- the four Aristotelian relations (relative to a logical system S):

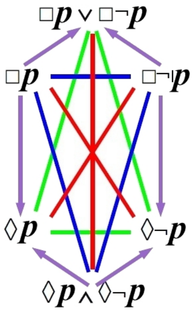
  φ and ψ are said to be

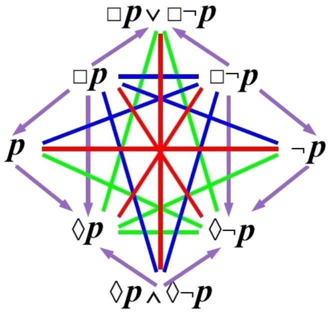
contradictory	iff	S$\models \neg(\varphi \wedge \psi)$	and	S$\models \neg(\neg\varphi \wedge \neg\psi)$
contrary	iff	S$\models \neg(\varphi \wedge \psi)$	and	S$\not\models \neg(\neg\varphi \wedge \neg\psi)$
subcontrary	iff	S$\not\models \neg(\varphi \wedge \psi)$	and	S$\models \neg(\neg\varphi \wedge \neg\psi)$
in subalternation	iff	S$\models \varphi \rightarrow \psi$	and	S$\not\models \psi \rightarrow \varphi$

(assumption: S has classical negation, conjunction, implication)

- throughout history: several proposals to extend the square
  - more formulas, more relations
  - larger and more complex diagrams
  - hexagons, octagons, cubes and other three-dimensional figures...







- the square and its extensions: hexagon, octagon, RDH, . . .
- the extensions are very interesting
  - well-motivated (singular propositions, Boolean closure)
  - throughout history (Sherwood hexagon, Buridan octagon)
  - interrelations (e.g. 3 squares inside JSB hexagon)

- yet:
  - (nearly) all logicians know about the square
  - (nearly) no logicians know about its extensions

- our explanation: “the Aristotelian square is very informative”
  - this claim sounds intuitive, but is also vague
  - provide precise and well-motivated framework

- recall the Aristotelian geometry: φ and ψ are said to be

contradictory	iff	$S \models \neg(\varphi \wedge \psi)$	and	$S \models \neg(\neg\varphi \wedge \neg\psi)$
contrary	iff	$S \models \neg(\varphi \wedge \psi)$	and	$S \not\models \neg(\neg\varphi \wedge \neg\psi)$
subcontrary	iff	$S \not\models \neg(\varphi \wedge \psi)$	and	$S \models \neg(\neg\varphi \wedge \neg\psi)$
in subalternation	iff	$S \models \varphi \to \psi$	and	$S \not\models \psi \to \varphi$

- problems with the Aristotelian geometry:
  - not mutually exclusive: e.g. $\bot$ and p are contrary *and* subaltern (problem disappears if we restrict to contingent formulas)
  - not exhaustive: e.g. p and $\Diamond p \wedge \Diamond\neg p$ are in no Arist. relation at all (if φ is contingent, then φ is in no Arist. relation to itself)
  - conceptual confusion: true/false together vs truth propagation
    - 'together' $\rightsquigarrow$ symmetrical relations (undirected)
    - 'propagation' $\rightsquigarrow$ asymmetrical relations (directed)

- the **Opposition Geometry (OG)**: φ and ψ are

contradictory	iff	$S \models \neg(\varphi \wedge \psi)$	and	$S \models \neg(\neg\varphi \wedge \neg\psi)$
contrary	iff	$S \models \neg(\varphi \wedge \psi)$	and	$S \not\models \neg(\neg\varphi \wedge \neg\psi)$
subcontrary	iff	$S \not\models \neg(\varphi \wedge \psi)$	and	$S \models \neg(\neg\varphi \wedge \neg\psi)$
non-contradictory	iff	$S \not\models \neg(\varphi \wedge \psi)$	and	$S \not\models \neg(\neg\varphi \wedge \neg\psi)$

- the **Implication Geometry (IG)**: φ and ψ are in

bi-implication	iff	$S \models \varphi \rightarrow \psi$	and	$S \models \psi \rightarrow \varphi$
left-implication	iff	$S \models \varphi \rightarrow \psi$	and	$S \not\models \psi \rightarrow \varphi$
right-implication	iff	$S \not\models \varphi \rightarrow \psi$	and	$S \models \psi \rightarrow \varphi$
non-implication	iff	$S \not\models \varphi \rightarrow \psi$	and	$S \not\models \psi \rightarrow \varphi$

- opposition relations: being true/false together $\varphi \wedge \psi$ and $\neg\varphi \wedge \neg\psi$
- implication relations: truth propagation $\varphi \wedge \neg\psi$ and $\neg\varphi \wedge \psi$

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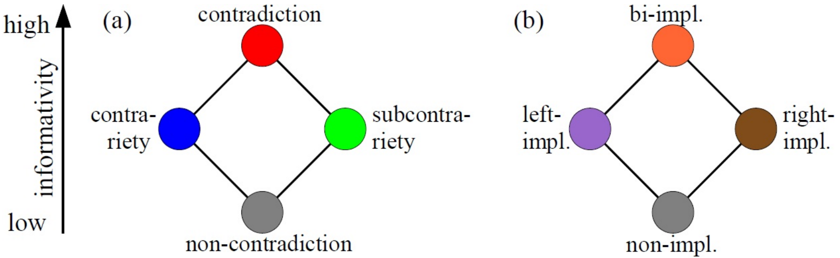
- OG and IG jointly solve the problems of the Aristotelian geometry:
  - each pair of formulas stands in exactly one opposition relation
  - each pair of formulas stands in exactly one implication relation
  - no more conceptual confusion

- conceptual independence, yet clear relationship (symmetry breaking):

$$\begin{array}{lcl} \mathsf{CD}(\varphi,\psi) & \Leftrightarrow & \mathsf{BI}(\psi,\neg\varphi) \\ \mathsf{C}(\varphi,\psi) & \Leftrightarrow & \mathsf{LI}(\psi,\neg\varphi) \\ \mathsf{SC}(\varphi,\psi) & \Leftrightarrow & \mathsf{RI}(\psi,\neg\varphi) \\ \mathsf{NCD}(\varphi,\psi) & \Leftrightarrow & \mathsf{NI}(\psi,\neg\varphi) \end{array}$$

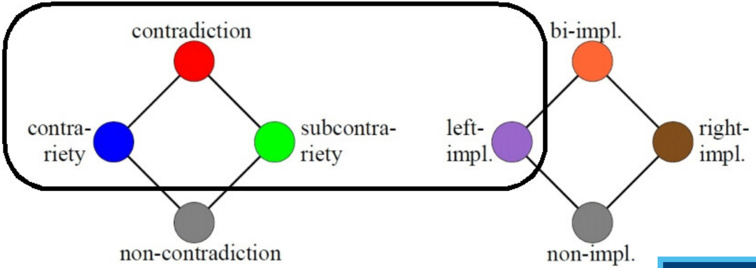
- Correia: two philosophical traditions
  - square as a theory of negation — commentaries on *De Interpretatione*
  - square as a theory of consequence — commentaries on *Prior Analytics*

- informativity of a relation holding between φ and ψ is inversely correlated with the number of states (models) it is compatible with
- informativity of the opposition and implication relations:



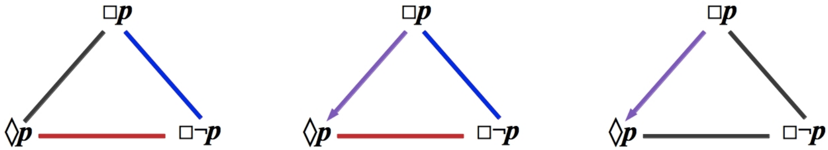
- close match between formal account and intuitions:
  - e.g. CD is more informative than C
  - if φ is known,
    - announcing CD(φ, ψ) uniquely determines ψ
    - announcing C(φ, ψ) doesn't uniquely determine ψ
- combinatorial results on finite Boolean algebras
  - Boolean algebra $\mathbb{B}_n$ with 2^n formulas, formula of level i:
    - 1 contradictory
    - $2^{n-i} - 1$ contraries and $2^i - 1$ subcontraries
    - $(2^{n-i} - 1)(2^i - 1)$ non-contradictories
  - $1 < 2^{n-i} - 1, 2^i - 1 < (2^{n-i} - 1)(2^i - 1)$ iff $1 < i < n - 1$
- coherent with earlier results:
  - opposition and implication yield isomorphic informativity lattices
  - CD(φ, ψ) $\Leftrightarrow$ BI($\psi, \neg\varphi$), ...

- why is the Aristotelian square special? Because it is very informative *diagram* in a very informative *geometry*
- Aristotelian geometry: hybrid between
  - opposition geometry: contradiction, contrariety, subcontrariety
  - implication geometry: left-implication (subalternation)
- these relations are highly informative (in their geometries)

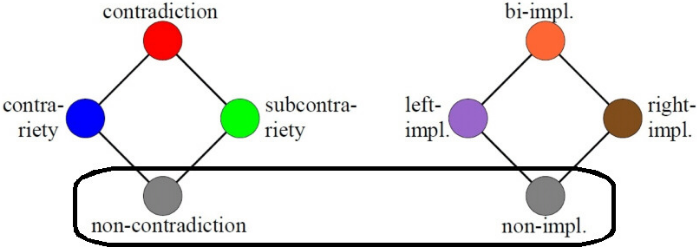


- given any two formulas:
  - they stand in exactly one opposition relation R
  - they stand in exactly one implication relation S

- if R is strictly more informative than S, then R is Aristotelian
- if S is strictly more informative than R, then S is Aristotelian
  - example 1: $\Box p$ and $\Diamond p$: non-contradiction and **left-implication**
  - example 2: $\Box p$ and $\Box\neg p$: **contrariety** and non-implication
  - example 3: $\Diamond p$ and $\Box\neg p$: **contradiction** and non-implication



- given any two formulas:
  - they stand in exactly one opposition relation R
  - they stand in exactly one implication relation S
- what if neither relation is strictly more informative than the other?
- theorem: this can only occur in one case: NCD + NI (unconnectedness)



- Aristotelian gap = information gap
  - no Aristotelian relation at all (non-exhaustiveness of AG)
  - combination of the two *least informative* relations

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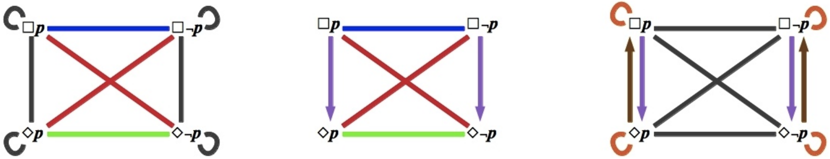
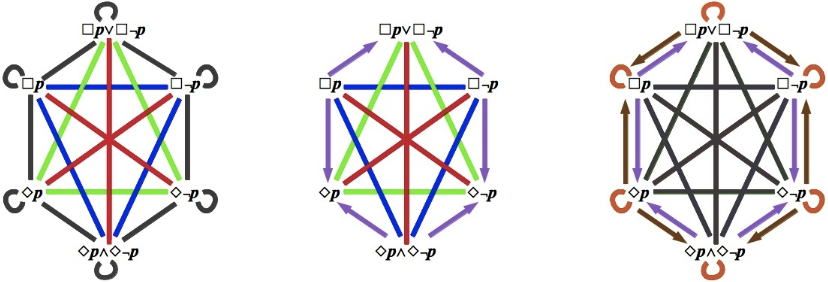
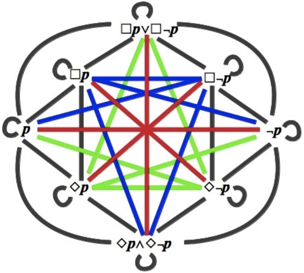
- no unconnectedness in the classical Aristotelian square

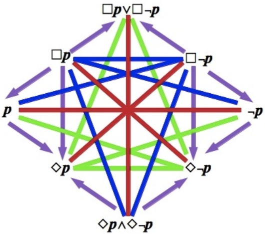


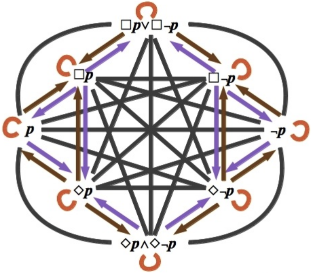
- no unconnectedness in the Jacoby-Sesmat-Blanché hexagon



- unconnectedness in the Béziau octagon
- e.g. p and $\Diamond p \wedge \Diamond \neg p$ are unconnected







- logical geometry: Aristotelian square of oppositions and its extensions
- the Aristotelian square is highly informative:
  - Aristotelian geometry is hybrid: *maximize* informativity
    $\Rightarrow$ applies to *all* Aristotelian diagrams
  - avoid unconnectedness: *minimize* uninformativity
    $\Rightarrow$ some Aristotelian diagrams succeed better than others
    - classical square, JSB hexagon, SC hexagon don't have unconnectedness
    - Béziau octagon (and many other diagrams) do have unconnectedness
- Q: what about the JSB hexagon, SC hexagon, etc.?
  - equally informative as the square
  - yet less widely known. . .
- A: requires yet another geometry: **duality**

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Aristotelian diagrams are (highly) **context-sensitive/logic-sensitive**:

- by virtue of the Aristotelian relations themselves:
  - two formulas may be contradictory in S_1 (e.g. $many_1/few_1$) but contrary in S_2 (e.g. $many_2/few_2$)
  - two formulas may be in subalternation in S_1 (e.g. SYL) but unconnected in S_2 (e.g. FOL)
- by virtue of the convention that Aristotelian diagrams only contain contingent formulas: two formulas may be tautological/contradictory in S_1 but not in S_2.
- by virtue of the convention that Aristotelian diagrams only contain formulas up to logical equivalence: two formulas may be equivalent in S_1 but not in S_2 (e.g. $\Box\Box p$ and $\Box p$ are in subalternation in T but equivalent in S4).

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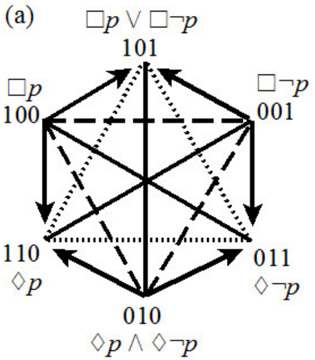
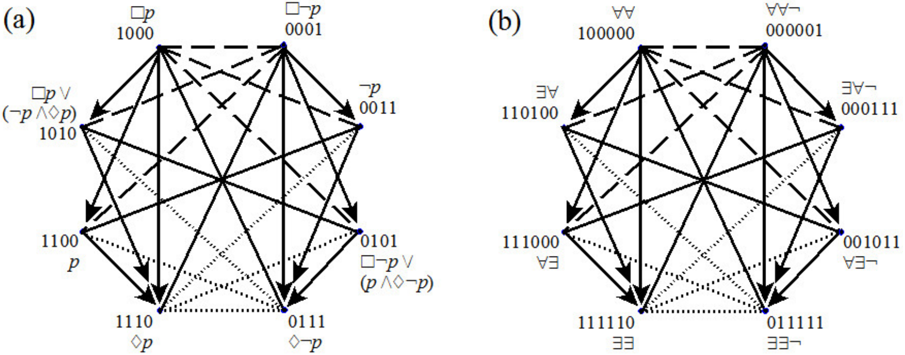
Two fragments which have an isomorphic Aristotelian structure may

- nevertheless not be isomorphic from a Boolean point of view.
- require an encoding with bitstrings of different length

Strong versus **weak** JSB hexagons (Pellissier, 2008)



Buridan octagons for S5 versus "combined operators":



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Two propositions are:

***contradictory* (CD)**	iff	they cannot be true together and they cannot be false together,
***contrary* (C)**	iff	they cannot be true together but they can be false together,
***subcontrary* (SC)**	iff	they can be true together but they cannot be false together,
***in subalternation* (SA)**	iff	the first proposition entails the second but the second doesn't entail the first

The set of Aristotelian relations is fundamentally *hybrid*:

- CD, C and SC are symmetric; definition $\sim$ being true/false together
  SA is not symmetric; definition $\sim$ truth propagation.
- CD is a functional relation, but C, SC and SA are not.
- Smessaert & Demey (2014)

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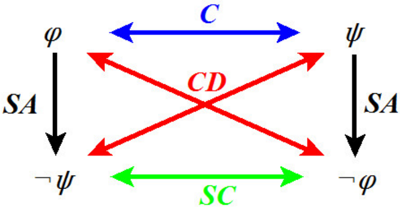
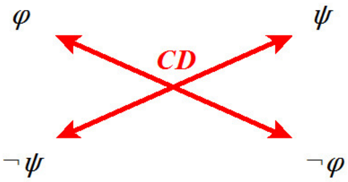
Any fragment of 4 formulas from a logical language $\mathcal{L}$ for a logical system S which is closed under negation (i.e. which consists of two pairs of contradictories) yields an *Aristotelian square* which is

$$\begin{aligned} \textbf{classical} &\equiv (2 \times \text{CD}) + (2 \times \text{SA}) + (1 \times \text{C}) + (1 \times \text{SC}) \\ \textbf{degenerate} &\equiv (2 \times \text{CD}) \end{aligned}$$



classical Aristotelian square



degenerate Aristotelian square

The n-ary connectives/operators O_1 and O_2 are one another's:

external negation **(EN)** iff for all $\varphi_1, \ldots, \varphi_n$
$O_2(\varphi_1, \ldots, \varphi_n) \equiv \neg O_1(\varphi_1, \ldots, \varphi_n)$

internal negation **(IN)** iff for all $\varphi_1, \ldots, \varphi_n$
$O_2(\varphi_1, \ldots, \varphi_n) \equiv O_1(\neg\varphi_1, \ldots, \neg\varphi_n)$

dual negation **(DN)** iff for all $\varphi_1, \ldots, \varphi_n$
$O_2(\varphi_1, \ldots, \varphi_n) \equiv \neg O_1(\neg\varphi_1, \ldots, \neg\varphi_n)$

Transpose definitions of EN/IN/DN from *operators* to *formulas*: if operators O_1 and O_2 are each other's EN/IN/DN, then formulas $O_1(\varphi_1 \ldots \varphi_n)$ and $O_2(\varphi_1 \ldots \varphi_n)$ are said to be each other's EN/IN/DN as well.

The set of duality relations is fundamentally *uniform*:

- EN, IN and DN are all symmetric relations.
- EN, IN and DN are all functional relations.

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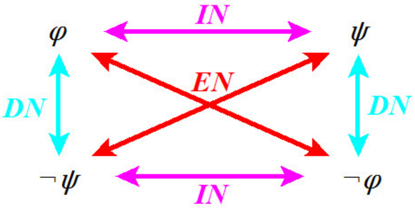
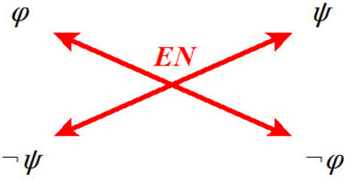
Any fragment of 4 formulas from a logical language $\mathcal{L}$ for a logical system S which is closed under negation (i.e. which consists of two pairs of contradictories) yields a *duality square* which is

$$\begin{aligned} \textbf{classical} &\equiv (2 \times \text{EN}) + (2 \times \text{IN}) + (2 \times \text{DN}) \\ \textbf{degenerate} &\equiv (2 \times \text{EN}) \end{aligned}$$

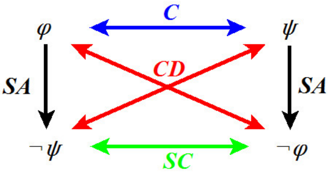


classical duality square

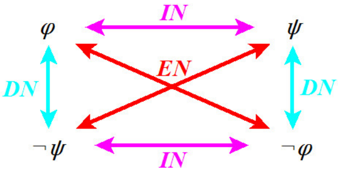


degenerate duality square

- Löbner (1990,2011), Peters & Westerståhl (2006), Westerståhl (2012), Demey (2012), Smessaert (2012).
- All duality relations are symmetric but not all Aristotelian relations are.
- All duality relations are functional but not all Aristotelian relations are.
- The duality relation IN corresponds to Aristotelian C and/or SC.
- Aristotelian relations are highly logic-sensitive, whereas duality relations are insensitive to underlying logic: Demey (2015), Demey & Smessaert (2016).



classical Aristotelian square



classical duality square

The functions ID, ENEG, INEG and DUAL jointly form a group that is isomorphic to the *Klein four group* $\mathbf{V}_4$. Its Cayley table looks as follows:

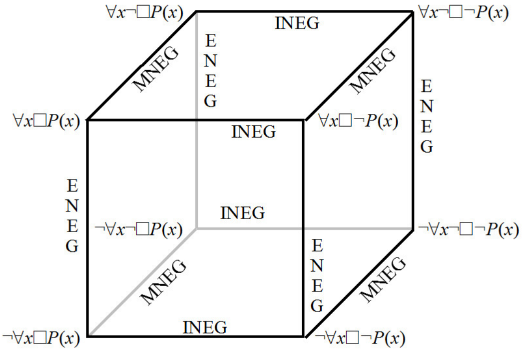
$\circ$	ID	ENEG	INEG	DUAL
ID	ID	ENEG	INEG	DUAL
ENEG	ENEG	ID	DUAL	INEG
INEG	INEG	DUAL	ID	ENEG
DUAL	DUAL	INEG	ENEG	ID

$\mathbf{V}_4$ is isomorphic to the direct product of $\mathbb{Z}_2$ with itself, i.e. $\mathbf{V}_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. The Cayley table for $\mathbb{Z}_2 \times \mathbb{Z}_2$ looks as follows:

$\circ$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(0,0)$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(1,0)$	$(1,0)$	$(0,0)$	$(1,1)$	$(0,1)$
$(0,1)$	$(0,1)$	$(1,1)$	$(0,0)$	$(1,0)$
$(1,1)$	$(1,1)$	$(0,1)$	$(1,0)$	$(0,0)$

generalisation

- from duality square to duality cube
- from 2 negation positions to 3 negation positions
- from $\mathbb{Z}_2 \times \mathbb{Z}_2$ to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$



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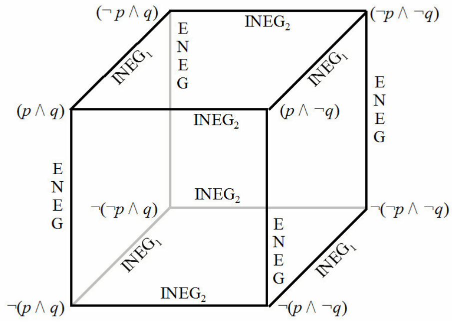
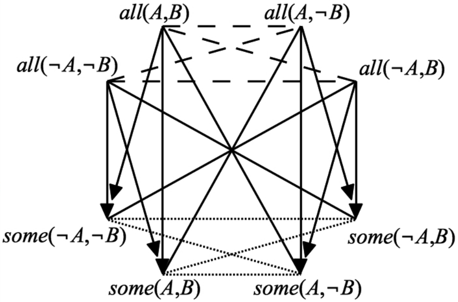
Generalized Post-duality

- with propositional connectives
- in the Keynes-Johnson octagon with subject negation





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