Visualising the Boolean Algebra $\mathbb{B}_4$ in 3D

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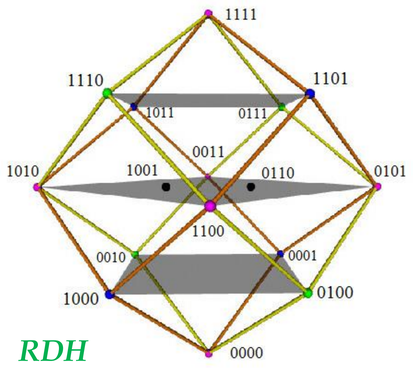
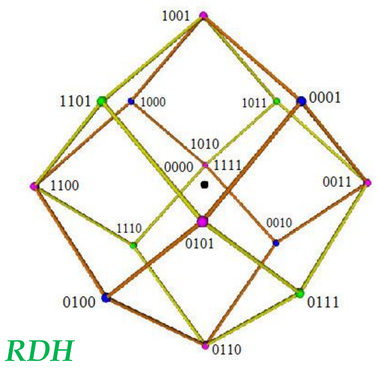
Rhombic Dodecahedron (RDH)



RDH
Hasse perspective



RDH
Aristotelian perspective

RDH =
12 rhombic faces +
14 vertices + 24 edges
dual of cuboctahedron

(Smessaert 2009;
Smessaert & Demey 2014/2015)

LOGICAL GEOMETRY

systematic study of Aristotelian and related logical diagrams

- abstract-logical properties
- visual-geometrical properties

visual differences significantly influence user comprehension

informational equivalence ⟺ **computational equivalence**

case study: the Boolean algebra $\mathbb{B}_4$

a single logical structure
Boolean algebra $\mathbb{B}_4 = \{0,1\}^4$
bitstrings of length 4 (0000,…)

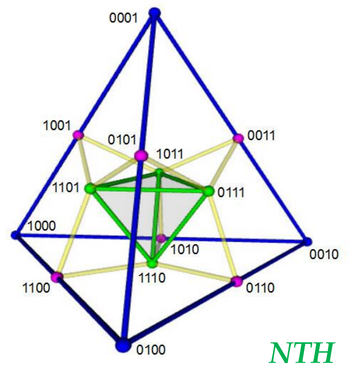
⟺

2 different 3D visualisations
- the rhombic dodecahedron
- the nested tetrahedron

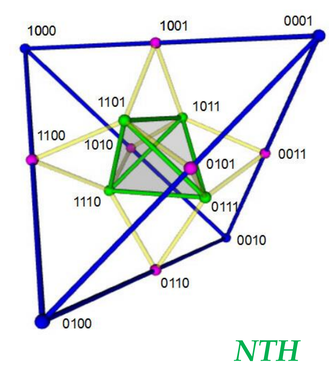
Aristotelian relations in $\mathbb{B}_4$: bitstrings b_1 and b_2 are:

contradictory (CD) iff $b_1 \wedge b_2 = 0000$ and $b_1 \vee b_2 = 1111$
contrary (C) iff $b_1 \wedge b_2 = 0000$ and $b_1 \vee b_2 \neq 1111$
subcontrary (SC) iff $b_1 \wedge b_2 \neq 0000$ and $b_1 \vee b_2 = 1111$
in ***subalternation*** (SA) iff $b_1 \wedge b_2 = b_1$ and $b_1 \vee b_2 \neq b_1$

Nested Tetrahedron (NTH)



NTH
vertex-face perspective

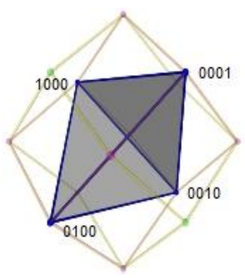


NTH
edge perspective

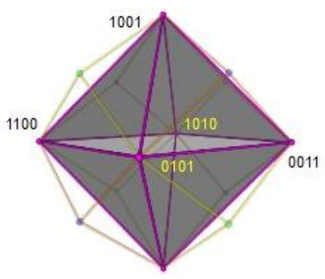
NTH =
4 triangular faces +
4 vertices + 6 edges
self-dual => "nested"

(Dubois & Prade 2012,; Ciucci e.a. 2012; Lewis Carroll/Moretti 2014)

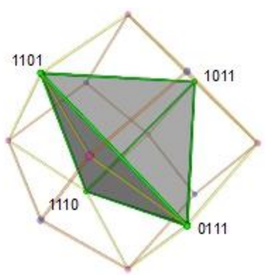
RDH



level 1



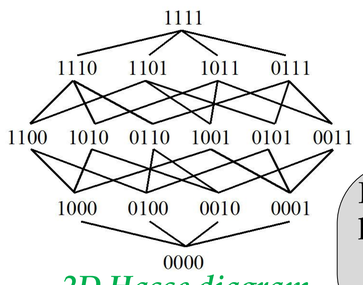
level 2



level 3

levels are ***not*** parallel hyperplanes
levels are ***not*** geometrical dimensions
⇓
RDH is ***not*** level-preserving at all

REPRESENTING LEVELS



2D Hasse diagram

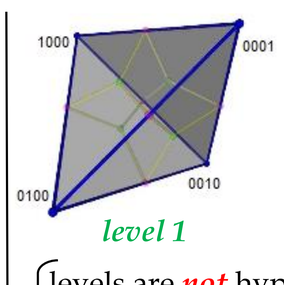
bitstring **level L** ≡
number of bit positions
with value 1
(1100=L2; 1101=L3, ...)

Hasse diagram levels =
horizontal hyperplanes
orthogonal to vertical
implication direction

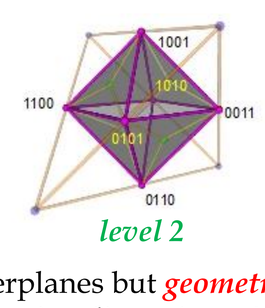
structure of visualisation
≈
represented logical structure
(Tversky 2005)

Congruence Principle

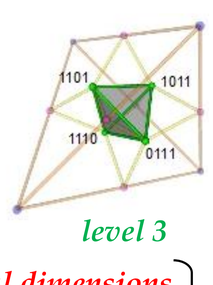
NTH



level 1



level 2

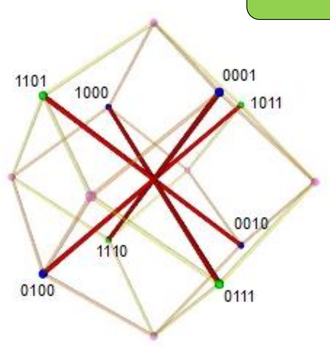


level 3

levels are ***not*** hyperplanes but ***geometrical dimensions***
L1 ≈ *zero*-dimensionality = 4 *vertices*
L2 ≈ *one*-dimensionality = **midpoints** of 6 *edges*
L3 ≈ *two*-dimensionality = **midpoints** of 4 *faces*
⇓
NTH is level-preserving

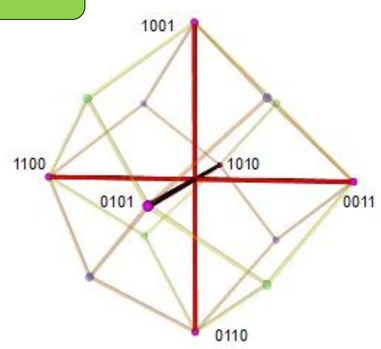
NTH observes the Congruence Principle much better than RDH w.r.t. representing levels

RDH



L1-L3 contradiction

central symmetry
and
maximal distance



L2-L2 contradiction

central symmetry
and
maximal distance

REPRESENTING CONTRADICTION

contradiction relation =
symmetric and **functional**
⇓
Aristotelian diagrams usually have **central symmetry**

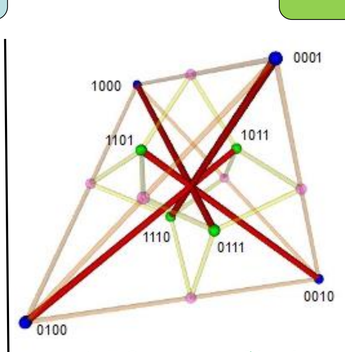
contradictory bitstrings are located at:
- diametrically opposed vertices
- at the same distance from the diagram's centre

contradiction relation =
strongest opposition relation
(bitstring contradiction =
switching values in **all** bit positions)
⇓
"*maximal logical distance*"
visualised by means of
"*maximal geometrical distance*"

(vertex representing bitstring *b* is farthest removed from vertex representing bitstring ¬*b*)

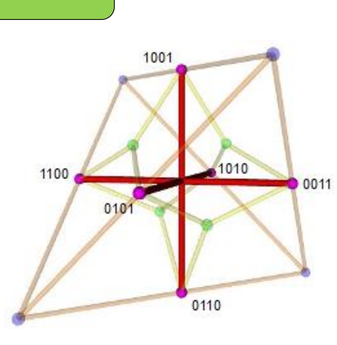
Congruence Principle

NTH



L1-L3 contradiction

no central symmetry
and
no maximal distance

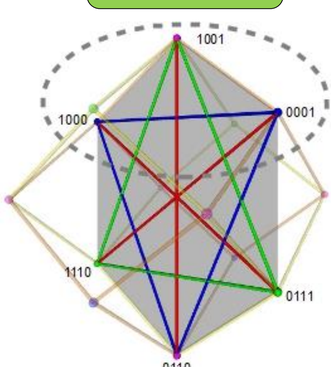


L2-L2 contradiction

central symmetry
but
no maximal distance

RDH observes the Congruence Principle much better than NTH w.r.t. representing contradiction

RDH

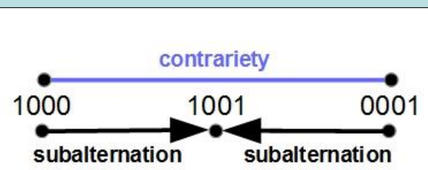


JSB-hexagon

no triples of collinear vertices
⇓
distinct lines for:
- 1000-0001 contrariety
- 1000-1001 subalternation
- 0001-1001 subalternation

⇓
no overlap/coincidence of visual components

REPRESENTING THE IMPLICATION/OPPOSITION INTERPLAY



The structure and content of the visualisation should be readily and accurately perceived and comprehended
(Tversky 2005)

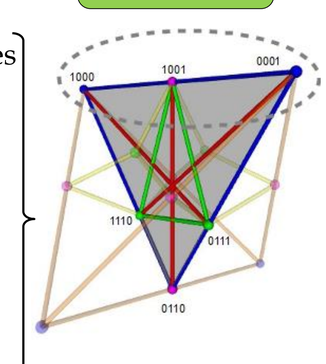
3 distinct Aristotelian relations of opposition and implication
⇓
3 distinct visual components

Apprehension Principle

NTH

triples of collinear vertices
⇓
no distinct lines for:
- 1000-0001 contrariety and 1000-1001 subalternation
- 1000-0001 contrariety and 0001-1001 subalternation

⇓
overlap/coincidence of visual components



JSB-hexagon

RDH observes the Apprehension Principle much better than NTH w.r.t. representing the implication/opposition interplay

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