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Introduction to Logical Geometry

5. Case Studies and Philosophical Outlook

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1. Basic Concepts, Bitstring Semantics and (Iso)morphisms
2. Abstract-Logical Properties of Aristotelian Diagrams, Part I
   ☞ Aristotelian, Opposition, Implication and Duality Relations
3. Visual-Geometric Properties of Aristotelian Diagrams
   ☞ Informational Equivalence, Cognition, Symmetry and Distance
4. Abstract-Logical Properties of Aristotelian Diagrams, Part II
   ☞ Boolean Structure and Logic-Sensitivity
5. **Case Studies and Philosophical Outlook**

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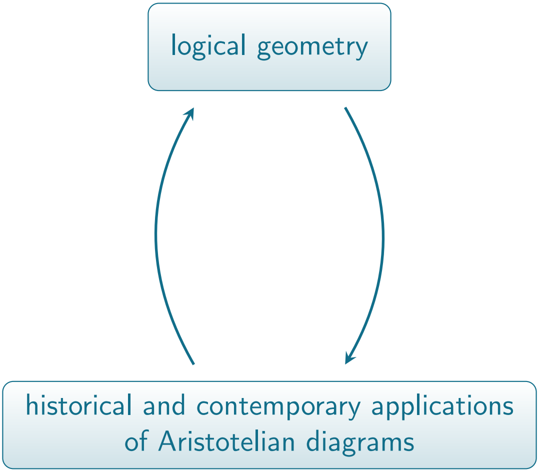
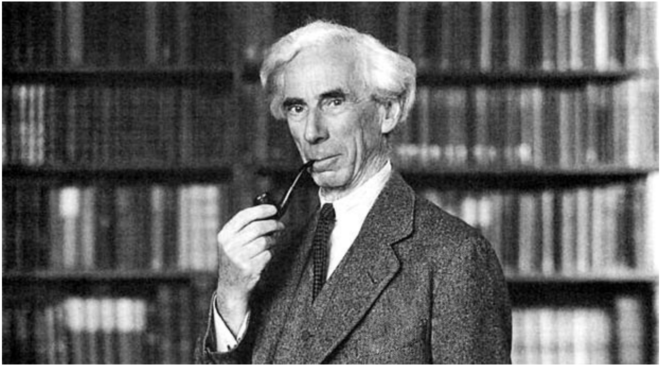

logical geometry
historical and contemporary applications
of Aristotelian diagrams



"ever since the beginning of the seventeenth century, almost every serious intellectual advance has had to begin with an attack on some Aristotelian doctrine; in logic, this is still true at the present day"

- definite descriptions in natural language:
  - the president of the United States
  - the man standing over there
  - the so-and-so

- they can occur in
  - **subject position** e.g. The president was diagnosed with Covid-19.
  - predicate position e.g. Joe Biden is currently the president.

- Russell's quantificational analysis of 'the A is B'

$$\exists x\Big(Ax \wedge \forall y(Ay \rightarrow y = x) \wedge Bx\Big)$$

- Neale's restricted quantifier notation

$$[\text{the } x\colon Ax]Bx$$

- $[\text{the } x\colon Ax]Bx \quad \equiv_{\mathsf{FOL}} \quad (\textsc{ex}) \wedge (\textsc{un}) \wedge (\textsc{uv})$

(EX)	$\exists x Ax$	there exists at least one A
(UN)	$\forall x \forall y ((Ax \wedge Ay) \rightarrow x = y)$	there exists at most one A
(UV)	$\forall x (Ax \rightarrow Bx)$	all As are B

- much of the subsequent literature on Russell's quantificational theory of definite descriptions has focused on one of these three conditions

- [the x: Ax]Bx $\equiv_{\mathsf{FOL}}$ $(\textsc{ex}) \wedge (\textsc{un}) \wedge (\textsc{uv})$

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- what is the linguistic status of (EX)?
  - Russell: (EX) is part of the **truth conditions** of 'the A is B'
    $\Rightarrow$ if (EX) is false, then 'the A is B' is false
  - Strawson: (EX) is a **presupposition** of 'the A is B'
    $\Rightarrow$ if (EX) is false, then 'the A is B' does not have a truth value at all

- [the x: Ax]Bx $\equiv_{\mathsf{FOL}}$ $(\textsc{ex}) \wedge (\textsc{un}) \wedge (\textsc{uv})$

(EX)	$\exists x Ax$	there exists at least one A
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- the problem of **incomplete definite descriptions** (for which (UN) fails) e.g. the book is on the shelf $\Rightarrow$ there is at most one book in the universe

- refinements and alternatives:
  - ellipsis theories (Vendler)
  - quantifier domain restriction theories (Stanley and Szabó)
  - pragmatic theories (Heim, Szabó)

- [the x: Ax]Bx $\equiv_{\text{FOL}}$ (EX) $\wedge$ (UN) $\wedge$ (UV)

  (EX) $\exists x Ax$ — there exists at least one A
  (UN) $\forall x \forall y((Ax \wedge Ay) \rightarrow x = y)$ — there exists at most one A
  (UV) $\forall x(Ax \rightarrow Bx)$ — all As are B

- much of the subsequent literature on Russell's quantificational theory of definite descriptions has focused on one of these three conditions

- what about **non-singular** definite descriptions?
  - plurals — e.g. The wives of King Henry VIII were pale.
  - mass nouns — e.g. The water in the Dead Sea is very salty.
- such descriptions also satisfy a version of (UV) (Sharvy, Brogaard)

An Aristotelian square for definite descriptions

- Russell: what is the negation of ‘the A is B’?
  - law of excluded middle $\Rightarrow$ ‘the A is B’ is true or ‘the A is not B’ is true
  - but if there are no As, then both statements seem to be false
- Russell: ‘the A is not B’ is **ambiguous** (scope)
  - $\neg\exists x\Big(Ax \wedge \forall y(Ay \rightarrow y = x) \wedge Bx\Big)$ $\neg$[the x: Ax]Bx
  - $\exists x\Big(Ax \wedge \forall y(Ay \rightarrow y = x) \wedge \neg Bx\Big)$ [the x: Ax]$\neg Bx$
- first interpretation:
  - Boolean negation of ‘the A is B’
  - if there are no As, then [the x: Ax]Bx is false, $\neg$[the x: Ax]Bx is true
- second interpretation:
  - if there are no As, then [the x: Ax]Bx and [the x: Ax]$\neg Bx$ are false
  - not the Boolean negation of ‘the A is B’

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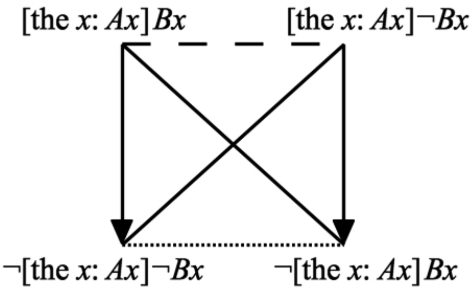
- crucial insight: the two interpretations of 'the A is not B' distinguished by Russell stand in different Aristotelian relations to 'the A is B'
  - $[\text{the } x\colon Ax]Bx$ and $\neg[\text{the } x\colon Ax]Bx$ are FOL-contradictory
  - $[\text{the } x\colon Ax]Bx$ and $[\text{the } x\colon Ax]\neg Bx$ are FOL-contrary
- cf. Haack (1978), Speranza and Horn (2010, 2012), Martin (2016)
- natural move: consider a **fourth formula** (with two negations)

$$\begin{array}{rr} \exists x\big(Ax \wedge \forall y(Ay \rightarrow y = x) \wedge Bx\big) & [\text{the } x\colon Ax]Bx \\ \neg\exists x\big(Ax \wedge \forall y(Ay \rightarrow y = x) \wedge Bx\big) & \neg[\text{the } x\colon Ax]Bx \\ \exists x\big(Ax \wedge \forall y(Ay \rightarrow y = x) \wedge \neg Bx\big) & [\text{the } x\colon Ax]\neg Bx \\ \neg\exists x\big(Ax \wedge \forall y(Ay \rightarrow y = x) \wedge \neg Bx\big) & \neg[\text{the } x\colon Ax]\neg Bx \end{array}$$

- consider the fragment $\mathcal{F}_{dd}$ containing these 4 formulas
- $(\mathcal{F}_{dd}, \text{FOL})$ is a **classical square**

An Aristotelian square for definite descriptions

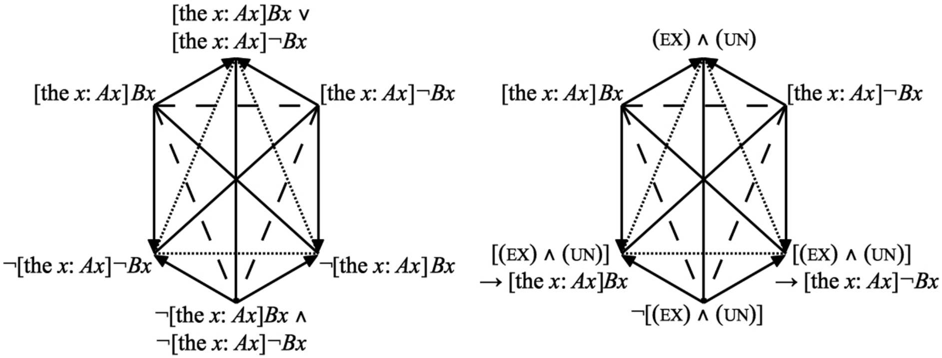


- this is an **Aristotelian** square
- but also a **duality** square

☞ lecture 2

- this square is fully defined in 'ordinary' FOL $\Rightarrow$ acceptable for Russell
- summarizes Russell's solution to puzzle on law of excluded middle
- interesting new formula: $\neg[\text{the } x\colon Ax]\neg Bx$
  - expresses a weak version of 'the A is B'
    $\neg[\text{the } x\colon Ax]\neg Bx \quad \equiv_{\mathsf{FOL}} \quad [(\textsc{ex}) \wedge (\textsc{un})] \rightarrow [\text{the } x\colon Ax]Bx$
    - if there is exactly one A,
      $[\text{the } x\colon Ax]Bx$ and $\neg[\text{the } x\colon Ax]\neg Bx$ always have the same truth value
    - in all other cases,
      $[\text{the } x\colon Ax]Bx$ is always false, whereas $\neg[\text{the } x\colon Ax]\neg Bx$ is always true
  - self-predication principles: what is the logical status of 'the A is A'?
    - $[\text{the } x\colon Ax]Ax$ is not a FOL-tautology
    - $\neg[\text{the } x\colon Ax]\neg Ax$ is a FOL-tautology

- the Aristotelian square for definite descriptions is not Boolean closed
- its Boolean closure is a **JSB hexagon**
- importance of the (EX)- and (UN)-conditions



- the definite description formulas induce the partition $\Pi_{\mathsf{FOL}}(\mathcal{F}_{dd}) := \{\alpha_1,\ \alpha_2,\ \alpha_3\}$
  - $\alpha_1 :=$ [the x: Ax]Bx
  - $\alpha_2 :=$ [the x: Ax]$\neg Bx$
  - $\alpha_3 := \neg[(\textsc{ex}) \wedge (\textsc{un})]$

- example bitstring representations:
  - [the x: Ax]$Bx \equiv_{\mathsf{FOL}} \alpha_1$ $\rightsquigarrow$ gets represented as 100
  - $\neg$[the x: Ax]$\neg Bx \equiv_{\mathsf{FOL}} \alpha_1 \vee \alpha_3$ $\rightsquigarrow$ gets represented as 101

- logical perspective: the Boolean closure of the square/hexagon has $2^3 - 2 = 6$ contingent formulas

- conceptual/linguistic perspective: **recursive partitioning of logical space**

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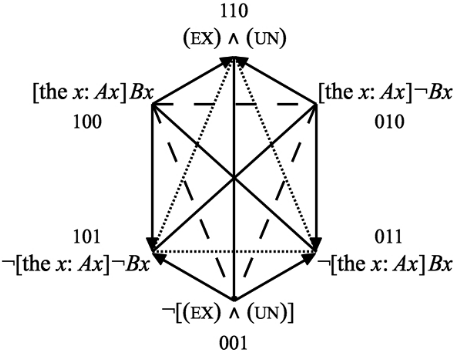
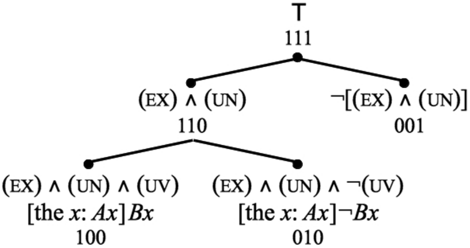

110
(EX) ∧ (UN)
[the x: Ax]Bx
100
[the x: Ax]¬Bx
010
101
¬[the x: Ax]¬Bx
011
¬[the x: Ax]Bx
¬[(EX) ∧ (UN)]
001

- view $\Pi_{\mathsf{FOL}}(\mathcal{F}_{dd})$ as the result of a process of **recursively partitioning and restricting logical space** (Seuren, Jaspers, Roelandt)
  - divide the logical universe: $(\textsc{ex}) \wedge (\textsc{un})$ vs. $\neg[(\textsc{ex}) \wedge (\textsc{un})]$
  - focus on the logical subuniverse defined by $(\textsc{ex}) \wedge (\textsc{un})$
  - recursively divide this subuniverse: [the x: $Ax]Bx$ vs. [the x: $Ax]\neg Bx$



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- another look at the ambiguity pointed out by Russell
  - 'the A is B' unambiguously corresponds to [the $x\colon Ax]Bx = 100$
  - relative to the entire universe, its negation is $\neg$[the $x\colon Ax]Bx = 011$
  - relative to the subuniverse ($\mathbf{11}0$), its negation is [the $x\colon Ax]\neg Bx = 010$

  $\Rightarrow$ Russell's two interpretations of 'the A is not B' correspond to negations of 'the A is B' **relative to two different universes** (semantic instead of syntactic characterization)

- Seuren and Jaspers's (2014) defeasible Principle of Complement Selection: "Natural complement selection is primarily relative to the proximate subuniverse, but there are overriding factors."

- overriding factors: intonation, additional linguistic material (Horn 1989)
  - *the* largest prime is not even; in fact, there doesn't *exist* a largest prime
  - *the* prime divisor of 30 is not even; in fact, 30 has *multiple* prime divisors

The categorical statements

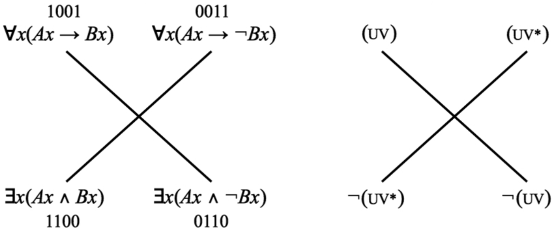
- consider the fragment $\mathcal{F}_{cat}$ of categorical statements from syllogistics:

A	all As are B	$\forall x(Ax \to Bx)$
I	some As are B	$\exists x(Ax \wedge Bx)$
E	no As are B	$\forall x(Ax \to \neg Bx)$
O	some As are not B	$\exists x(Ax \wedge \neg Bx)$

- already implicit in the definite description formulas
  - $[\text{the } x : Ax] \; Bx \equiv_{\text{FOL}} (\text{EX}) \wedge (\text{UN}) \wedge (\text{UV})$
  - $\neg[\text{the } x : Ax] \; Bx \equiv_{\text{FOL}} \neg(\text{EX}) \vee \neg(\text{UN}) \vee \neg(\text{UV})$
  - $[\text{the } x : Ax] \neg Bx \equiv_{\text{FOL}} (\text{EX}) \wedge (\text{UN}) \wedge (\text{UV}^*)$
  - $\neg[\text{the } x : Ax] \neg Bx \equiv_{\text{FOL}} \neg(\text{EX}) \vee \neg(\text{UN}) \vee \neg(\text{UV}^*)$

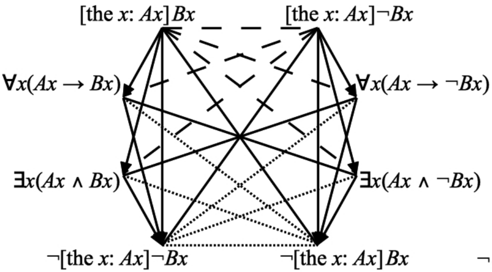
$$(\text{UV}) \equiv_{\text{FOL}} \forall x(Ax \to Bx) = \text{A}$$
$$\neg(\text{UV}) \equiv_{\text{FOL}} \exists x(Ax \wedge \neg Bx) = \text{O}$$
$$(\text{UV}^*) \equiv_{\text{FOL}} \forall x(Ax \to \neg Bx) = \text{E}$$
$$\neg(\text{UV}^*) \equiv_{\text{FOL}} \exists x(Ax \wedge Bx) = \text{I}$$

- first-order logic (FOL) has no existential import
- $\mathcal{F}_{cat}$ induces the partition $\Pi_{\mathsf{FOL}}(\mathcal{F}_{cat}) = \{\beta_1,\ \beta_2,\ \beta_3,\ \beta_4\}$:
  - $\beta_1 := \exists x Ax \wedge \forall x(Ax \rightarrow Bx)$
  - $\beta_2 := \exists x(Ax \wedge Bx) \wedge \exists x(Ax \wedge \neg Bx)$
  - $\beta_3 := \exists x Ax \wedge \forall x(Ax \rightarrow \neg Bx)$
  - $\beta_4 := \neg \exists x Ax$ (recursive partitioning)
- in FOL, the categorical statements constitute a **degenerate square**

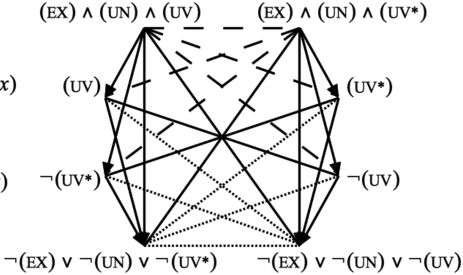


- there is a subalternation from [the x: Ax]Bx to the A-statement
- there is a subalternation from [the x: Ax]Bx to the I-statement
- and so on. . .
- summary:

  the interaction between the definite description formulas and the categorical statements gives rise to a **Buridan octagon**



[the x: Ax]Bx
[the x: Ax]¬Bx
∀x(Ax → Bx)
∀x(Ax → ¬Bx)
∃x(Ax ∧ Bx)
∃x(Ax ∧ ¬Bx)
¬[the x: Ax]¬Bx
¬[the x: Ax]Bx



(EX) ∧ (UN) ∧ (UV)
(EX) ∧ (UN) ∧ (UV*)
(UV)
(UV*)
¬(UV*)
¬(UV)
¬(EX) ∨ ¬(UN) ∨ ¬(UV*)
¬(EX) ∨ ¬(UN) ∨ ¬(UV)

- the definite descriptions induce the 3-partition $\Pi_{\mathsf{FOL}}(\mathcal{F}_{dd})$
- the categorical statements induce the 4-partition $\Pi_{\mathsf{FOL}}(\mathcal{F}_{cat})$

$\Rightarrow$ together, $\mathcal{F}_{ddcat} := \mathcal{F}_{dd} \cup \mathcal{F}_{cat}$ induces the 6-partition $\Pi_{\mathsf{FOL}}(\mathcal{F}_{ddcat}) = \Pi_{\mathsf{FOL}}(\mathcal{F}_{dd}) \wedge_{\mathsf{FOL}} \Pi_{\mathsf{FOL}}(\mathcal{F}_{cat})$

  - $\gamma_1 := \exists x \exists y (Ax \wedge Ay \wedge x \neq y) \wedge \forall x (Ax \rightarrow Bx)$
  - $\gamma_2 := \exists x (Ax \wedge Bx) \wedge \exists x (Ax \wedge \neg Bx)$
  - $\gamma_3 := \exists x \exists y (Ax \wedge Ay \wedge x \neq y) \wedge \forall x (Ax \rightarrow \neg Bx)$
  - $\gamma_4 := [\text{the } x\colon Ax]Bx$
  - $\gamma_5 := [\text{the } x\colon Ax]\neg Bx$
  - $\gamma_6 := \neg \exists x Ax$

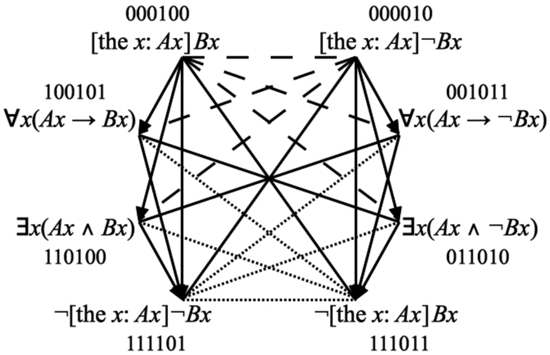
- $\Pi_{\mathsf{FOL}}(\mathcal{F}_{ddcat})$ is a refinement of $\Pi_{\mathsf{FOL}}(\mathcal{F}_{dd})$

  $\Rightarrow$ $\gamma_4 = \alpha_1$ and $\gamma_5 = \alpha_2$, while $\gamma_1 \vee \gamma_2 \vee \gamma_3 \vee \gamma_6 \equiv_{\mathsf{FOL}} \alpha_3$

- $\Pi_{\mathsf{FOL}}(\mathcal{F}_{ddcat})$ is a refinement of $\Pi_{\mathsf{FOL}}(\mathcal{F}_{cat})$

  $\Rightarrow$ $\gamma_2 = \beta_2$ and $\gamma_6 = \beta_4$, while $\gamma_1 \vee \gamma_4 \equiv_{\mathsf{FOL}} \beta_1$ and $\gamma_3 \vee \gamma_5 \equiv_{\mathsf{FOL}} \beta_3$

- $\Pi_{\mathsf{FOL}}(\mathcal{F}_{ddcat})$ allows us to encode every formula of the Buridan octagon



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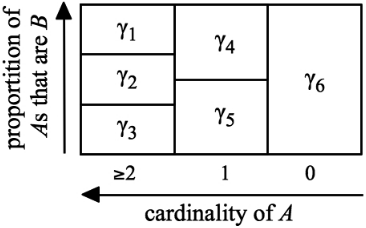
- $\Pi_{\mathsf{FOL}}(\mathcal{F}_{ddcat})$ is ordered along two semi-independent dimensions
  - the **cardinality** of (the extension of) A
  - the **proportion** of As that are B
- **semi**-independent: higher cardinalities allow for more fine-grained proportionality distinctions
- visual perspective on the refinement of partitions
  - $\Pi_{\mathsf{FOL}}(\mathcal{F}_{ddcat})$ is a refinement of $\Pi_{\mathsf{FOL}}(\mathcal{F}_{dd})$
  - $\alpha_1 \equiv_{\mathsf{FOL}} \gamma_4$ and $\alpha_2 \equiv_{\mathsf{FOL}} \gamma_5$ and $\alpha_3 \equiv_{\mathsf{FOL}} \gamma_1 \vee \gamma_2 \vee \gamma_3 \vee \gamma_6$



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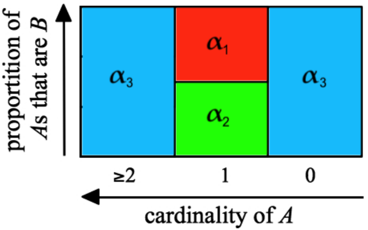

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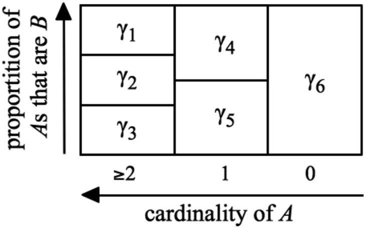
  - $\beta_1 \equiv_{\mathsf{FOL}} \gamma_1 \vee \gamma_4$ and $\beta_2 \equiv_{\mathsf{FOL}} \gamma_2$ and $\beta_3 \equiv_{\mathsf{FOL}} \gamma_3 \vee \gamma_5$ and $\beta_4 \equiv_{\mathsf{FOL}} \gamma_6$



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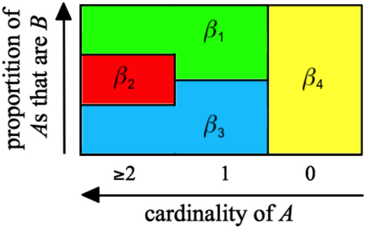
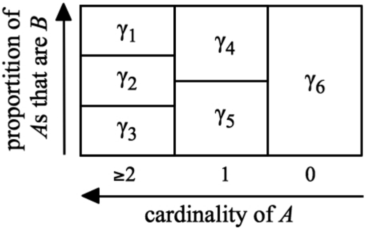


- $\Pi_{\mathsf{FOL}}(\mathcal{F}_{ddcat})$ is ordered along two semi-independent dimensions
  - the **cardinality** of (the extension of) A
  - the **proportion** of As that are B
- **semi**-independent: higher cardinalities allow for more fine-grained proportionality distinctions
- ongoing work on linguistic aspects:
  - plausible partitioning process?
  - split the '≥ 2'-region into '≥ 3'- and '$= 2$'-subregions ('both', 'neither')



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- recent work on existential import (Seuren, **Chatti and Schang**, Read)
- for every categorical statement $\varphi \in \mathcal{F}_{\mathit{cat}}$, define
  - variant $\varphi_{\text{imp!}}$ that explicitly **has** existential import $\exists x Ax \wedge \varphi$
  - variant $\varphi_{\text{imp?}}$ that explicitly **lacks** existential import $\exists x Ax \to \varphi$

$$
\begin{array}{llll}
\mathsf{A}_{\text{imp?}} & \equiv_{\mathsf{FOL}} \forall x(Ax \to Bx) & \equiv_{\mathsf{FOL}} & (\text{UV}) \\
\mathsf{I}_{\text{imp!}} & \equiv_{\mathsf{FOL}} \exists x(Ax \wedge Bx) & \equiv_{\mathsf{FOL}} & \neg(\text{UV}^*) \\
\mathsf{E}_{\text{imp?}} & \equiv_{\mathsf{FOL}} \forall x(Ax \to \neg Bx) & \equiv_{\mathsf{FOL}} & (\text{UV}^*) \\
\mathsf{O}_{\text{imp!}} & \equiv_{\mathsf{FOL}} \exists x(Ax \wedge \neg Bx) & \equiv_{\mathsf{FOL}} & \neg(\text{UV}) \\
\mathsf{A}_{\text{imp!}} & \equiv_{\mathsf{FOL}} \exists x Ax \wedge \forall x(Ax \to Bx) & \equiv_{\mathsf{FOL}} & (\text{EX}) \wedge (\text{UV}) \\
\mathsf{I}_{\text{imp?}} & \equiv_{\mathsf{FOL}} \exists x Ax \to \exists x(Ax \wedge Bx) & \equiv_{\mathsf{FOL}} & \neg(\text{EX}) \vee \neg(\text{UV}^*) \\
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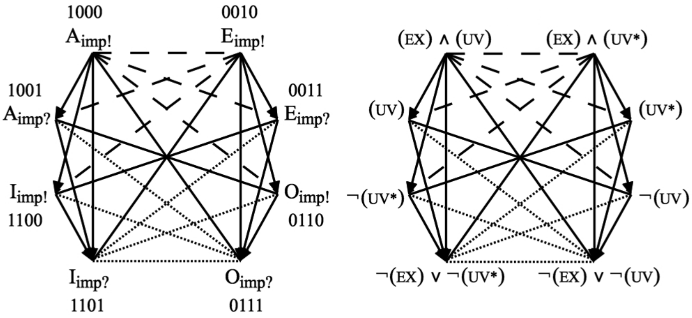
\end{array}
$$

- $\mathcal{F}^{?!}_{\mathit{cat}} := \{\varphi_{\text{imp?}}, \varphi_{\text{imp!}} \mid \varphi \in \mathcal{F}_{\mathit{cat}}\}$

A related octagon

- Chatti and Schang's $\mathcal{F}^{?!}_{cat}$ is closely related to our $\mathcal{F}_{ddcat}$ and $\mathcal{F}_{cat}$
- $(\mathcal{F}^{?!}_{cat}, \text{FOL})$ is a Buridan octagon, just like $(\mathcal{F}_{ddcat}, \text{FOL})$
- $\Pi_{\text{FOL}}(\mathcal{F}^{?!}_{cat}) = \{\text{A}_{\text{imp!}}, \text{I}_{\text{imp!}} \wedge \text{O}_{\text{imp!}}, \text{E}_{\text{imp!}}, \neg\exists x Ax\} = \Pi_{\text{FOL}}(\mathcal{F}_{cat})$



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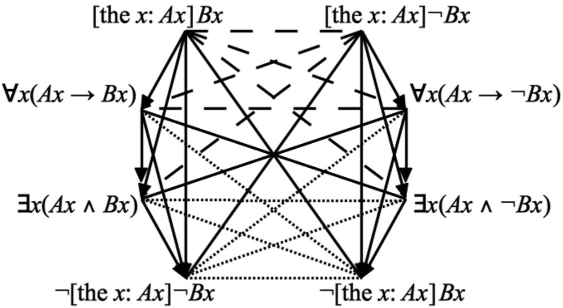
- Buridan octagon $(\mathcal{F}_{ddcat}, \mathsf{FOL})$
  - induces the partition $\Pi_{\mathsf{FOL}}(\mathcal{F}_{ddcat})$, with 6 anchor formulas
  - $[\text{the } x\colon Ax]\ \ Bx \not\equiv_{\mathsf{FOL}} \mathsf{A} \wedge \mathsf{I}$ $(000100 \neq 100101 \wedge 110100)$
  - $\neg[\text{the } x\colon Ax]\neg Bx \not\equiv_{\mathsf{FOL}} \mathsf{A} \vee \mathsf{I}$ $(111101 \neq 100101 \vee 110100)$

- Buridan octagon $(\mathcal{F}^{?!}_{cat}, \mathsf{FOL})$
  - induces the partition $\Pi_{\mathsf{FOL}}(\mathcal{F}_{cat})$, with 4 anchor formulas
  - $\mathsf{A}_{imp!} \equiv_{\mathsf{FOL}} \mathsf{A}_{imp?} \wedge \mathsf{I}_{imp!}$ $(1000 = 1001 \wedge 1100)$
  - $\mathsf{I}_{imp?} \equiv_{\mathsf{FOL}} \mathsf{A}_{imp?} \vee \mathsf{I}_{imp!}$ $(1101 = 1001 \wedge 1100)$

- summary:
  - one and the **same Aristotelian family** (Buridan octagons)
  - **different Boolean subtypes** ☞ lecture 4

- until now: only worked in ordinary first-order logic (FOL)
- Chatti and Schang: deal with existential import by adding $(\neg)\exists x Ax$ as conjunct/disjunct to the categorical statements
- alternative approach:
  - existential import $\neq$ property of **individual formulas**
  - existential import $=$ property of **underlying logical system**
- introduce new logical system SYL:
  - SYL = FOL + $\exists x Ax$
  - interpreted on FOL-models $\langle D, I \rangle$ such that $I(A) \neq \emptyset$
  - analogy with modal logic:
    - KD = K + $\Diamond\top$
    - interpreted on serial frames,
      i.e. K-frames $\langle W, R \rangle$ such that $R[w] \neq \emptyset$ (for all $w \in W$)

- move from FOL to SYL
- influence on the categorical statements:
  - e.g. A and E are unconnected in FOL, but become contrary in SYL, etc.
  - the degen. square $(\mathcal{F}_{cat}, \text{FOL})$ turns into a classical square $(\mathcal{F}_{cat}, \text{SYL})$
- no influence on the definite description formulas:
  - e.g. $[\text{the } x\colon Ax]Bx$ and $[\text{the } x\colon Ax]\neg Bx$ are contrary in FOL, and remain so in SYL
  - the classical square $(\mathcal{F}_{dd}, \text{FOL})$ remains a classical square $(\mathcal{F}_{dd}, \text{SYL})$
- no influence on the interaction between definite descriptions and categorical statements:
  - e.g. subalternation from $[\text{the } x\colon Ax]Bx$ to A and to I in FOL, and this remains so in SYL
- from Buridan octagon $(\mathcal{F}_{ddcat}, \text{FOL})$ to **Lenzen octagon** $(\mathcal{F}_{ddcat}, \text{SYL})$

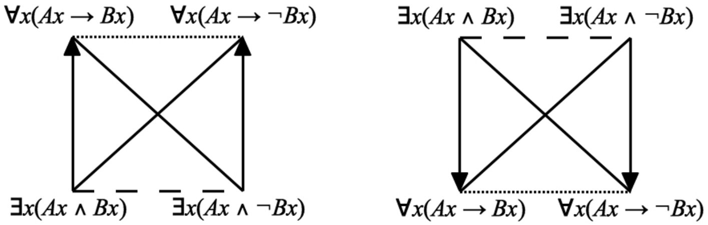


[the x: Ax]Bx
[the x: Ax]¬Bx
∀x(Ax → Bx)
∀x(Ax → ¬Bx)
∃x(Ax ∧ Bx)
∃x(Ax ∧ ¬Bx)
¬[the x: Ax]¬Bx
¬[the x: Ax]Bx

- which partition $\Pi_{\mathsf{SYL}}(\mathcal{F}_{ddcat})$ is induced?
  - SYL is a stronger logical system than FOL
  - consider the anchor formula $\neg\exists x Ax = \gamma_6 \in \Pi_{\mathsf{FOL}}(\mathcal{F}_{ddcat})$: FOL-consistent, but SYL-inconsistent
  - $\Pi_{\mathsf{SYL}}(\mathcal{F}_{ddcat}) = \Pi_{\mathsf{FOL}}(\mathcal{F}_{ddcat}) - \{\gamma_6\}$
- deleting the sixth bit position $\Rightarrow$ unified perspective on all changes:
  - A (100101) and E (001011) go from FOL-unconnected to SYL-contrary
  - I (110100) and O (011010) go from FOL-unconnected to SYL-subcontr.
  - A (100101) and I (110100) go from FOL-unconnected to SYL-subaltern
  - $[\text{the } x\colon Ax]Bx$ (000100) and $[\text{the } x\colon Ax]Bx$ (000010) are FOL-contrary, and remain so in SYL
  - $[\text{the } x\colon Ax]Bx$ (000100) and A (100101) are FOL-subaltern, and remain so in SYL

The role of uniqueness

- (EX) and (UN) play complementary roles in Russell's theory
- introduce new logical system SYL*
  - SYL* = FOL + $\forall x \forall y((Ax \wedge Ay) \to x = y)$
  - interpreted on FOL-models $\langle D, I \rangle$ such that $|I(A)| \leq 1$
- move from FOL to SYL*
- no influence on the definite description formulas
  - e.g. [the x: Ax]Bx and [the x: Ax]$\neg Bx$ are contrary in FOL, and remain so in SYL*
  - the classical square $(\mathcal{F}_{dd}, \text{FOL})$ remains a classical square $(\mathcal{F}_{dd}, \text{SYL}^*)$
- influence on the categorical statements:
  - e.g. A and E are unconnected in FOL, but become subcontrary in SYL*
  - the degen. square $(\mathcal{F}_{cat}, \text{FOL})$ turns into a **classical square** $(\mathcal{F}_{cat}, \text{SYL}^*)$
  - note: this classical square is 'flipped upside down'!

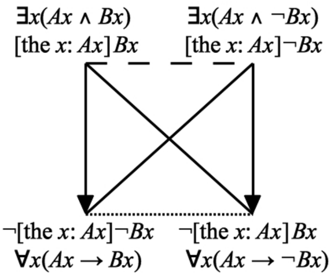


- example: take A to be the predicate 'monarch of country C'
- then always $|I(A)| \leq 1$
  - if C is a monarchy, then $|I(A)| = 1$
  - if C is a republic, then $|I(A)| = 0$

- move from FOL to SYL*
- influence on the interaction between definite descriptions and categorical statements
  - e.g. [the $x\colon Ax]Bx$ and the E-statement go from FOL-contrary to SYL*-contradictory
  - e.g. in FOL there is a subalternation from [the $x\colon Ax]Bx$ to the I-statement, but in SYL* they are logically equivalent to each other
- **pairwise collapse** of dd. formulas and categorical statements:

$$\begin{array}{rcccl} [\text{the } x\colon Ax]Bx & \equiv_{\mathsf{SYL}^*} & \mathsf{I} & = & \exists x(Ax \wedge Bx) \\ \neg[\text{the } x\colon Ax]Bx & \equiv_{\mathsf{SYL}^*} & \mathsf{E} & = & \forall x(Ax \rightarrow \neg Bx) \\ [\text{the } x\colon Ax]\neg Bx & \equiv_{\mathsf{SYL}^*} & \mathsf{O} & = & \exists x(Ax \wedge \neg Bx) \\ \neg[\text{the } x\colon Ax]\neg Bx & \equiv_{\mathsf{SYL}^*} & \mathsf{A} & = & \forall x(Ax \rightarrow Bx) \end{array}$$

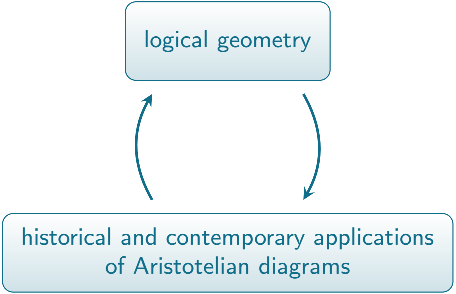
- from Buridan octagon $(\mathcal{F}_{ddcat}, \mathsf{FOL})$ to **collapsed (flipped) classical square** $(\mathcal{F}_{ddcat}, \mathsf{SYL}^*)$



∃x(Ax ∧ Bx)
[the x: Ax]Bx
∃x(Ax ∧ ¬Bx)
[the x: Ax]¬Bx
¬[the x: Ax]¬Bx
∀x(Ax → Bx)
¬[the x: Ax]Bx
∀x(Ax → ¬Bx)

- an elementary calculation yields the partition $\Pi_{\mathsf{SYL}^*}(\mathcal{F}_{ddcat})$ $= \{\exists x Ax \wedge \forall x(Ax \rightarrow Bx), \exists x Ax \wedge \forall x(Ax \rightarrow \neg Bx), \neg\exists x Ax\}$
- $\Pi_{\mathsf{SYL}^*}(\mathcal{F}_{ddcat}) = \Pi_{\mathsf{FOL}}(\mathcal{F}_{ddcat}) - \{\gamma_1, \gamma_2, \gamma_3\}$ (up to $\equiv_{\mathsf{SYL}^*}$)
  - SYL* is a stronger logical system than FOL
  - $\gamma_1, \gamma_2, \gamma_3$ are FOL-consistent, but SYL*-inconsistent
- $\Pi_{\mathsf{SYL}^*}(\mathcal{F}_{ddcat}) = \Pi_{\mathsf{FOL}}(\mathcal{F}_{dd})$ (up to $\equiv_{\mathsf{SYL}^*}$)
  - $\Pi_{\mathsf{FOL}}(\mathcal{F}_{dd})$ is the partition for the dd. square in FOL
  - moving from FOL to SYL* did not change this square
  - but did cause it to coincide with the categorical statement square
- $\Pi_{\mathsf{SYL}^*}(\mathcal{F}_{ddcat}) = \Pi_{\mathsf{FOL}}(\mathcal{F}_{cat}) - \{\beta_2\}$ (up to $\equiv_{\mathsf{SYL}^*}$)
  - $\Pi_{\mathsf{FOL}}(\mathcal{F}_{cat})$ is the partition for the cat. statement square in FOL
  - SYL* is stronger than FOL; β_2 is FOL-consistent, but SYL*-inconsistent
  - moving from FOL to SYL* triggered change from degen. square to (flipped) classical square, which coincides with the dd. square

- Aristotelian diagrams for Russell's theory of definite descriptions
  - classical square, JSB hexagon, Buridan octagon. . .
  - the formula $\neg[\text{the } x\colon Ax]\neg Bx$ and its interpretation, negations of $[\text{the } x\colon Ax]Bx$ relative to different subuniverses. . .
- phenomena and techniques studied in logical geometry
  - bitstring analysis, Boolean closure. . .
  - Boolean subtypes, logic-sensitivity. . .



1. Basic Concepts and Bitstring Semantics
2. Abstract-Logical Properties of Aristotelian Diagrams, Part I
   ☞ Aristotelian, Opposition, Implication and Duality Relations
3. Visual-Geometric Properties of Aristotelian Diagrams
   ☞ Informational Equivalence, Symmetry and Distance
4. Abstract-Logical Properties of Aristotelian Diagrams, Part II
   ☞ Boolean Structure and Logic-Sensitivity
5. Case Studies and **Philosophical Outlook**

- recall the guiding metaphor:
  - Aristotelian diagrams constitute a **language**
  - logical geometry is the **linguistics** that studies that language

- double motivation for logical geometry:
  - Aristotelian diagrams as **objects of independent interest**
  - Aristotelian diagrams as a **widely-used language**

- fundamental question:
  - **why** are Aristotelian diagrams used so widely to begin with?
  - **which reasons** do the authors themselves offer for their usage?

  (practice-based philosophy of logic)

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1. the received view: Aristotelian diagrams as **pedagogical devices**
2. the **multimodal** nature of Aristotelian diagrams
3. the **implicit normativity** of the tradition of using Aristotelian diagrams
4. Aristotelian diagrams as **heuristic tools**

- these explanations are **not mutually exclusive**
- Aristotelian diagrams as **technologies** or instruments
  - a technology can be created with one function in mind
  - and later acquire another function
  - the latter can even become the primary function

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- Aristotelian diagrams are mainly **pedagogical devices**
- visual nature ⇒ **mnemonic** value
- helpful to introduce novice students to the abstract discipline of logic
- Kruja et al., *History of Graph Drawing*, 2002:

  "Squares of opposition were pedagogical tools used in the teaching of logic ... They were designed to facilitate the recall of knowledge that students already had"
- Nicole Oresme, *Le livre du ciel et du monde*, 1377:

  "In order to illustrate this, I clarify it by means of a figure very similar to that used to introduce children to logic."

  (Et pour ce mieux entendre, je le desclaire en une figure presque semblable a une que l'en fait pour la premiere introduction des enfans en logique.)

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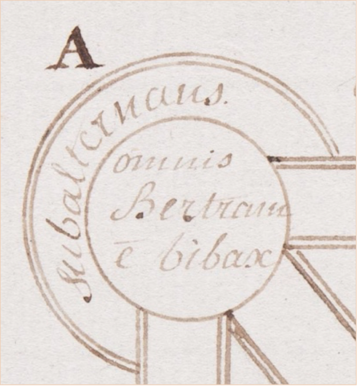
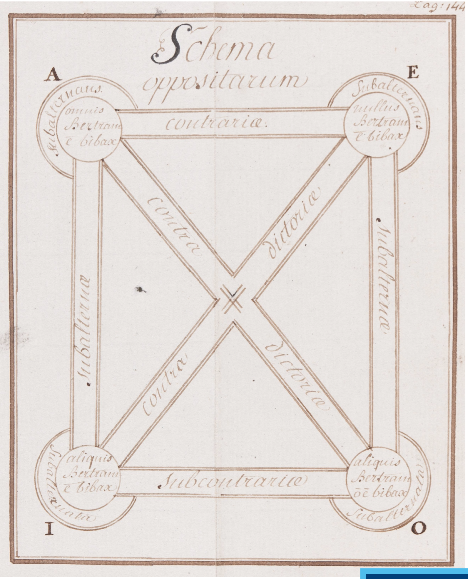
LIBRI SECVNDI TRACTATVS
omnis homo est animal
Contrariæ
nullus homo est animal
Subalternæ
Contradictoriæ
Subalternæ
quidam homo est animal
Subcontrariæ
quidam homo non est animal
De materia propositionum Ca. sextum.
De legibus propositionum. Ca. septimum

Joerden
Logik im Recht
Grundlagen
und Anwendungsbeispiele
Springer


A
Subalternans.
omnis
Bertram
ē bibax


Schema
oppositarum
A
E
I
O
contrariæ
subcontrariæ
contra
dictoriæ
Subalternæ
omnis
Bertram
ē bibax
nullus
Bertram
ē bibax
aliquis
Bertram
ē bibax
aliquis
Bertram
ōē bibax

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- the received view was accurate **in the past**:
  Aristotelian diagrams indeed were primarily/exclusively teaching tools
- but **today**, Aristotelian diagrams occur
  - not only in textbooks on logic
  - but mainly in **research-level** papers/monographs on **various disciplines** (logic, linguistics, psychology, computer science, etc.)

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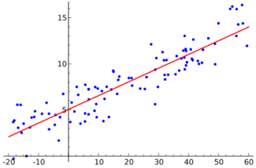
- Aristotelian diagrams offer cognitive advantages, because of their **multimodal** nature (visual + symbolic/textual)
- Aristotelian diagrams as a **visual summary** of some of the key properties of the logical system under investigation
- example: classical square of opposition for $(\mathcal{F}_{dd}, \mathsf{FOL})$
- analogy: graph vs. raw numeric data
- comparison with the received view (pedagogical devices):
  - both emphasize the **cognitive advantages** of Aristotelian diagrams
  - the second view accommodates **teaching and research** contexts



- Béziau, 2013:

  "The use of such a coloured diagram is very useful to understand in a direct, quick and synthetic way basic notions of modern logic, corresponding to the notion of Übersichtlichkeit [surveyability] that Wittgenstein was fond of"

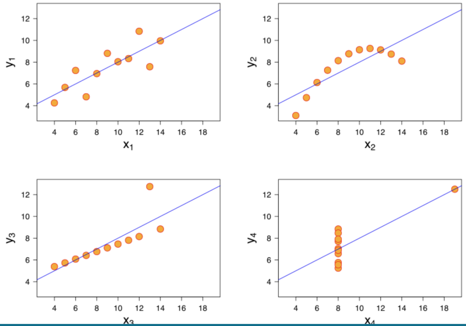
- Ciucci, Dubois & Prade, 2015:

  "Opposition structures are a powerful tool to express all properties of rough sets and fuzzy rough sets w.r.t. negation in a synthetic way."

- Eilenberg & Steenrod, 1952 (commutative diagrams in alg. topology):

  "The diagrams incorporate a large amount of information. Their use provides extensive savings in space and in mental effort."

- Aristotelian diagrams as a **visual summary** of a logical system
- is the emphasis on **visual** or on **summary**?
  put differently: how about non-visual summaries?
- analogy: graph vs. raw numeric data ⇒ **Anscombe's quartet**:
  - very different datasets, with very different graphs
  - yet (near-)identical summarizing statistics (mean, variance, correlation)



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- Anscombe's quartet — applied to logical geometry
  - very different . . . datasets — . . . logics/fragments
  - (near-)identical . . . statistics — . . . non-visual summaries
  - very different . . . graphs — . . . Aristotelian diagrams

- example:
  - $\mathcal{F}_{cat}$ in different logical systems SYL and SYL*
  - $|\Pi_{\mathsf{SYL}}(\mathcal{F}_{cat})| = 3 = |\Pi_{\mathsf{SYL}^*}(\mathcal{F}_{cat})|$
  - classical square vs. flipped classical square

- example:
  - different fragments $\mathcal{F}_{cat}$ and $\mathcal{F}_{cat}^{?!}$
  - $|\Pi_{\mathsf{FOL}}(\mathcal{F}_{cat})| = 4 = |\Pi_{\mathsf{FOL}}(\mathcal{F}_{cat}^{?!})|$
  - degenerate square vs. Buridan octagon

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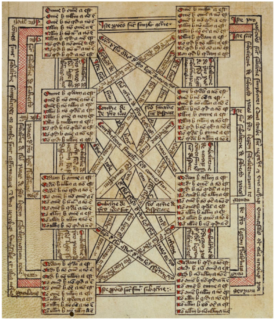
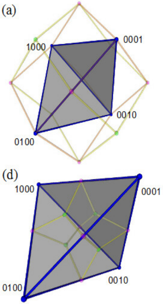
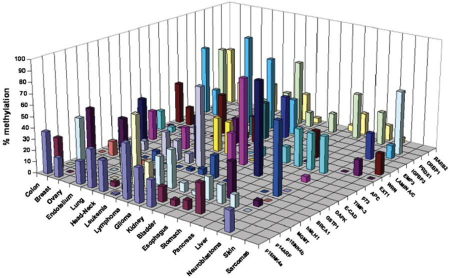
- the second view (multimodality) fits well with **visually 'simple'** diagrams, such as the square of opposition
- but what about **more visually complex** diagrams?







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- Aristotelian diagrams have a **very rich and respectable tradition** within the broader history of logic: many famous authors made use of these diagrams
- the tradition of using Aristotelian diagrams gets endowed with a kind of **(implicit) normativity** (tradition itself as object of reverence)
- Banerjee et al., 2018:

  "many artificial intelligence knowledge representation settings are sharing the same structures of opposition that extend or generalise the traditional square of opposition which dates back to Aristotle"
- Ciucci, 2016:

  "The study of oppositions starts in ancient Greece and has its main result in the Square of Opposition by Aristotle. In the last years, we can assist to a renewal of interest in this topic."

- this provides a (partial) explanation as to why we **continue** to use Aristotelian diagrams
- it takes the tradition of using Aristotelian diagrams as its starting point
- but how/why did this tradition **start** in the first place?

- Aristotelian diagrams as **heuristic tools**

- they enable researchers
  - to draw **high-level analogies** between seemingly unrelated frameworks
  - to introduce **new concepts** (by transferring them across frameworks)

- Aristotelian relations = ‘right’ layer of abstraction
  - not overly specific (otherwise, no analogies are possible)
  - not overly general (otherwise, the analogies become vacuous)

- Ciucci et al., 2014:
  The Structure of Oppositions in Rough Set Theory and Formal Concept Analysis - Toward a New Bridge between the Two Settings

- Dubois et al., 2015:
  The Cube of Opposition - A Structure underlying many Knowledge Representation Formalisms

- Read, 2012:
  "Buridan was able [. . .] to exhibit a strong analogy between modal, oblique and nonnormal propositions in his three octagons"

- think back of $\neg$[the x: Ax]$\neg Bx$ from the case study

- Yao, 2013:

  "With respect to the four logic expressions of the square of opposition, we can identify four subsets of attributes. [...] While the set of core attributes is well studied, the other [three] sets of attributes received much less attention."

Ongoing research: analogies and isomorphism

- Gentner's structure-mapping theory of analogy: analogy is a kind of isomorphism
- philosophy of representation systems (Barwise, Etchemendy, Hammer):
  - a good representation D is homomorphic to the represented application A
  - not all-or-nothing: degrees of homomorphicity
  - diagrams typically have the highest homomorphicity $\Rightarrow$ isomorphism

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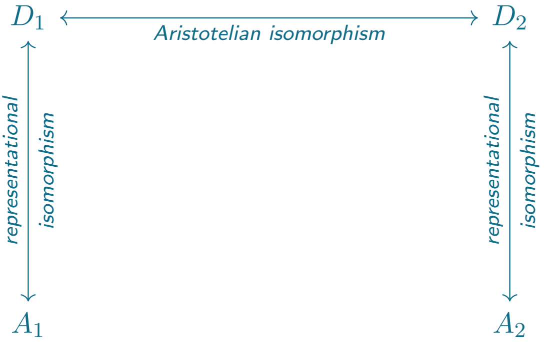

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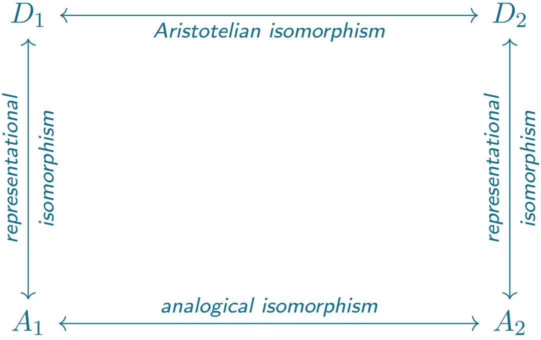

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Thank you! Questions?

More info: `www.logicalgeometry.org`