Introduction to Logical Geometry

4. Abstract-Logical Properties of Aristotelian Diagrams, Part II

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1. Basic Concepts, Bitstring Semantics and (Iso)morphisms
2. Abstract-Logical Properties of Aristotelian Diagrams, Part I
   ☞ Aristotelian, Opposition, Implication and Duality Relations
3. Visual-Geometric Properties of Aristotelian Diagrams
   ☞ Informational Equivalence, Cognition, Symmetry and Distance
4. **Abstract-Logical Properties of Aristotelian Diagrams, Part II**
   ☞ **Boolean Structure and Logic-Sensitivity**
5. Case Studies and Philosophical Outlook

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   ☞ **Boolean Structure** and Logic-Sensitivity

5. Case Studies and Philosophical Outlook

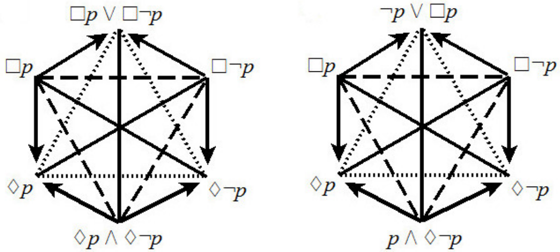
- recall from lecture 1:
  - since the Aristotelian relations are defined in purely Boolean terms, the Aristotelian structure of a fragment is entirely determined by its Boolean structure
  - if two fragments have the same Boolean structure, they also have the same Aristotelian structure
  - every Boolean isomorphism between two fragments is also an Aristotelian isomorphism between those fragments

- the inverse does **not** hold:
  - the Boolean structure of a fragment is not entirely determined by its Aristotelian structure
  - it is perfectly possible for two fragments to have the same Aristotelian structure, and yet different Boolean structures
  - there exist Aristotelian isomorphisms between two fragments that are not Boolean isomorphisms between those two fragments

- easiest + oldest example of this phenomenon (Pellissier 2008)
- two hexagons $D_\ell = (\mathcal{F}_\ell, \mathsf{S5})$ and $D_r = (\mathcal{F}_r, \mathsf{S5})$
- bijection $f\colon \mathcal{F}_\ell \to \mathcal{F}_r$:

  1. $f(\Box p) = \Box p$
  2. $f(\Diamond p) = \Diamond p$
  3. $f(\Box\neg p) = \Box\neg p$
  4. $f(\Diamond\neg p) = \Diamond\neg p$
  5. $f(\Box p \vee \Box\neg p) = \neg p \vee \Box p$
  6. $f(\Diamond p \wedge \Diamond\neg p) = p \wedge \Diamond\neg p$

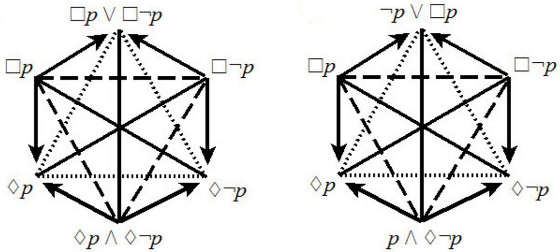
- $f\colon D_\ell \to D_r$ is clearly an Aristotelian isomorphism (check visually!)



- since D_ℓ and D_r are **Aristotelian isomorphic**, they belong to the **same Aristotelian family**
- specifically: both diagrams are Jacoby-Sesmat-Blanché (JSB) hexagons

- nevertheless, they have clear **Boolean differences**:

  (1) $\Box p \vee \Box\neg p$ is equivalent to the disjunction of $\Box p$ and $\Box\neg p$, but $\neg p \vee \Box p$ is not equivalent to the disjunction of $\Box p$ and $\Box\neg p$



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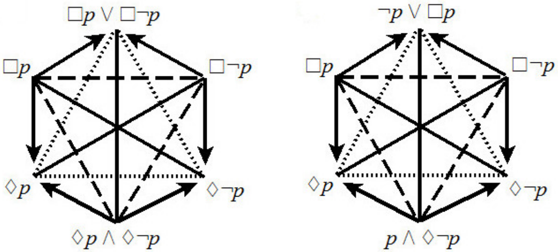
- specifically: both diagrams are Jacoby-Sesmat-Blanché (JSB) hexagons

- nevertheless, they have clear **Boolean differences**:

  (2) $\Diamond p \wedge \Diamond\neg p$ is equivalent to the conjunction of $\Diamond p$ and $\Diamond\neg p$, but $p \wedge \Diamond\neg p$ is not equivalent to the conjunction of $\Diamond p$ and $\Diamond\neg p$



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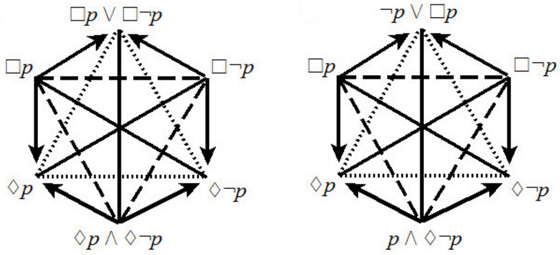
- since D_ℓ and D_r are **Aristotelian isomorphic**, they belong to the **same Aristotelian family**
- specifically: both diagrams are Jacoby-Sesmat-Blanché (JSB) hexagons
- nevertheless, they have clear **Boolean differences**:

  (3) the disjunction of $\Box p$, $\Box\neg p$ and $\Diamond p \wedge \Diamond\neg p$ is a tautology, but the disjunction of $\Box p$, $\Box\neg p$ and $p \wedge \Diamond\neg p$ is not a tautology



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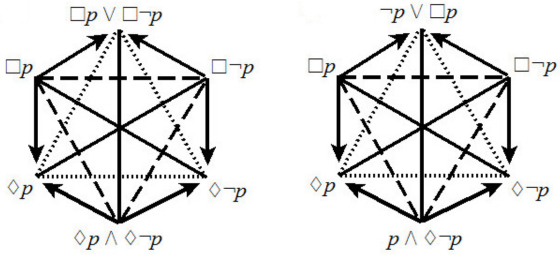
- since D_ℓ and D_r are **Aristotelian isomorphic**, they belong to the **same Aristotelian family**
- specifically: both diagrams are Jacoby-Sesmat-Blanché (JSB) hexagons

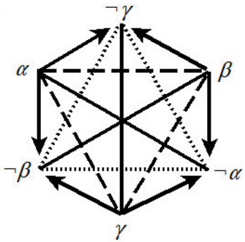
- nevertheless, they have clear **Boolean differences**:

  (4) the conjunction of $\Diamond p$, $\Diamond\neg p$ and $\Box p \vee \Box\neg p$ is a contradiction, but the conjunction of $\Diamond p$, $\Diamond\neg p$ and $\neg p \vee \Box p$ is not a contradiction



- **generic description** of a JSB hexagon: $(\mathcal{F}_{JSB}, \mathcal{R}_{JSB})$ (independent of concrete formulas, logical system, etc.)
  - $\mathcal{F}_{JSB} = \{\alpha, \beta, \gamma, \neg\alpha, \neg\beta, \neg\gamma\}$
  - $\mathcal{R}_{JSB}$ specifies the relations between the formulas of $\mathcal{F}_{JSB}$, e.g., the formulas α, β, γ are pairwise contrary

- the Aristotelian family of JSB hexagons has **two Boolean subtypes**:
  - **strong** JSB hexagon: $\alpha \vee \beta \vee \gamma$ is a tautology
  - **weak** JSB hexagon: $\alpha \vee \beta \vee \gamma$ is not a tautology



- consider different Boolean subtypes of some given Aristotelian family

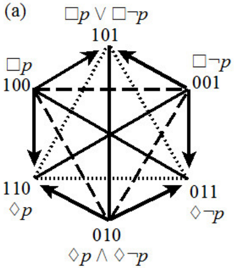
- **different Boolean subtypes** have **different Boolean properties**
- e.g. strong vs. weak JSB hexagon ⇒ at least 4 Boolean differences

- these differences can be summarized as follows:
  different Boolean subtypes have **different Boolean closures**,
- specifically: both diagrams are Jacoby-Sesmat-Blanché (JSB) hexagons

- recall that bitstring length measures the size of the Boolean closure

- **different Boolean subtypes** are encoded by means of **bitstrings of different lengths**

- our example of a **strong** JSB hexagon: $(\mathcal{F}_\ell, \mathsf{S5})$
  - induces the partition $\Pi_{\mathsf{S5}}(\mathcal{F}_\ell) := \{\Box p, \Diamond p \wedge \Diamond \neg p, \Box \neg p\}$
  - $|\Pi_{\mathsf{S5}}(\mathcal{F}_\ell)| = 3 \Rightarrow$ **bitstrings of length 3**
  - Boolean closure: $2^3 = 8$ elements, of which $2^3 - 2 = 6$ are contingent
- our example of a **weak** JSB hexagon: $(\mathcal{F}_r, \mathsf{S5})$
  - induces the partition $\Pi_{\mathsf{S5}}(\mathcal{F}_r) := \{\Box p, p \wedge \Diamond \neg p, \neg p \wedge \Diamond p, \Box \neg p\}$
  - $|\Pi_{\mathsf{S5}}(\mathcal{F}_r)| = 4 \Rightarrow$ **bitstrings of length 4**
  - Boolean closure: $2^4 = 16$ elements, of which $2^4 - 2 = 14$ are contingent



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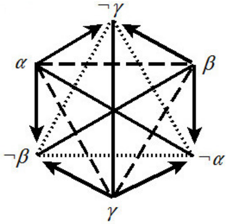
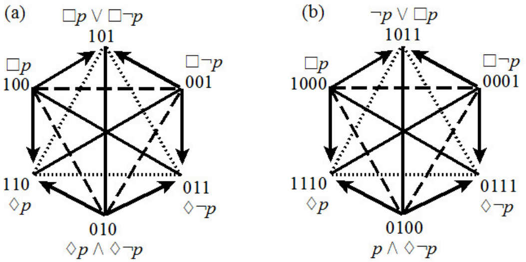
- this bitstring analysis summarizes all the individual Boolean differences

strong JSB	bitstrings of **length 3**	**weak**	bitstrings of **length 4**
$\neg\gamma \equiv \alpha \vee \beta$	$101 = 100 \vee 001$	$\not\equiv$	$1011 \neq 1000 \vee 0001$
$\gamma \equiv \neg\alpha \wedge \neg\beta$	$010 = 011 \wedge 110$	$\not\equiv$	$0100 \neq 0111 \wedge 1110$
$\alpha \vee \beta \vee \gamma \equiv \top$	$100 \vee 001 \vee 010 = 111$	$\not\equiv$	$1000 \vee 0001 \vee 0100 \neq 1111$
$\neg\alpha \wedge \neg\beta \wedge \neg\gamma \equiv \bot$	$011 \wedge 110 \wedge 101 = 000$	$\not\equiv$	$0111 \wedge 1110 \wedge 1011 \neq 0000$

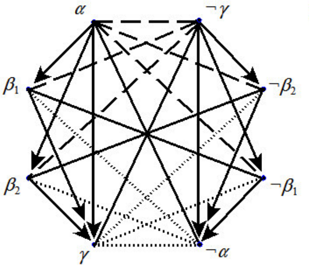




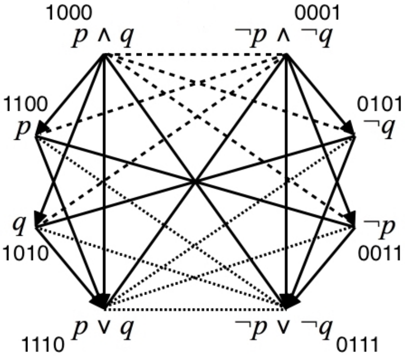
- generic description of a **Buridan octagon**: $(\mathcal{F}_{Buri}, \mathcal{R}_{Buri})$
  - $\mathcal{F}_{Buri} = \{\alpha, \beta_1, \beta_2, \gamma, \neg\alpha, \neg\beta_1, \neg\beta_2, \neg\gamma\}$
  - $\mathcal{R}_{Buri}$: subalternations from α to β_1, β_2 to γ; unconnected β_1, β_2

- the Buridan octagons come in **three Boolean subtypes**:

strong Buridan octagon	$\alpha \equiv \beta_1 \wedge \beta_2$	and	$\gamma \equiv \beta_1 \vee \beta_2$	**length 4**
intermediate Buridan octagon	$\alpha \equiv \beta_1 \wedge \beta_2$	XOR	$\gamma \equiv \beta_1 \vee \beta_2$	**length 5**
weak Buridan hexagon	$\alpha \equiv \beta_1 \wedge \beta_2$	nor	$\gamma \equiv \beta_1 \vee \beta_2$	**length 6**



- induces the partition $\Pi_{\mathsf{CPL}}(\mathcal{F}_{prop}) = \{p \wedge q, p \wedge \neg q, \neg p \wedge q, \neg p \wedge \neg q\}$
- 4 anchor formulas $\Rightarrow$ bitstrings of length 4
- $p \wedge q$ is equivalent to the conjunction of p and q $(1000 = 1100 \wedge 1010)$
- $p \vee q$ is equivalent to the disjunction of p and q $(1110 = 1100 \vee 1010)$



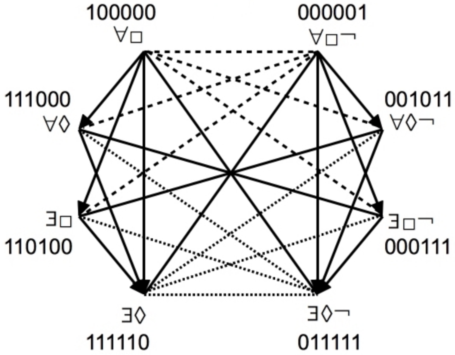
- fragment $\mathcal{F}_{modsyl}$ of 8 *de re* modal formulas (with ampliation):

1. all S are necessarily P	$\exists x\Diamond Sx \wedge \forall x(\Diamond Sx \to \Box Px)$	$\forall\Box$
2. all S are possibly P	$\exists x\Diamond Sx \wedge \forall x(\Diamond Sx \to \Diamond Px)$	$\forall\Diamond$
3. some S are necessarily P	$\exists x(\Diamond Sx \wedge \Box Px)$	$\exists\Box$
4. some S are possibly P	$\exists x(\Diamond Sx \wedge \Diamond Px)$	$\exists\Diamond$
5. all S are necessarily not P	$\forall x(\Diamond Sx \to \Box\neg Px)$	$\forall\Box\neg$
6. all S are possibly not P	$\forall x(\Diamond Sx \to \Diamond\neg Px)$	$\forall\Diamond\neg$
7. some S are necessarily not P	$\neg\exists x\Diamond Sx \vee \exists x(\Diamond Sx \wedge \Box\neg Px)$	$\exists\Box\neg$
8. some S are possibly not P	$\neg\exists x\Diamond Sx \vee \exists x(\Diamond Sx \wedge \Diamond\neg Px)$	$\exists\Diamond\neg$

- this induces the partition $\Pi_{\mathsf{FOS5}}(\mathcal{F}_{modsyl})$:

$$\begin{aligned}\{\quad & \forall\Box, \\ & \forall\Diamond \wedge \exists\Box \wedge \exists\Diamond\neg, \\ & \forall\Diamond \wedge \forall\Diamond\neg, \\ & \exists\Box \wedge \exists\Box\neg, \\ & \forall\Diamond\neg \wedge \exists\Box\neg \wedge \exists\Diamond, \\ & \forall\Box\neg \quad \}\end{aligned}$$

- 6 anchor formulas ⇒ bitstrings of length 6
- $\forall\Box \not\equiv \forall\Diamond \wedge \exists\Box$ $\qquad (100000 \neq 111000 \wedge 110100)$
- $\exists\Diamond \not\equiv \forall\Diamond \vee \exists\Box$ $\qquad (111110 \neq 111000 \vee 110100)$



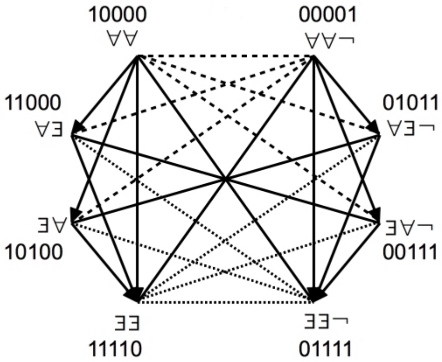
- fragment $\mathcal{F}_{unusual}$ of 8 propositions 'of unusual construction':

1. all S all P are	$\exists xSx \wedge \exists yPy \wedge \forall x(Sx \rightarrow \forall y(Py \rightarrow x = y))$	$\forall\forall$
2. all S some P are	$\exists xSx \wedge \forall x(Sx \rightarrow \exists y(Py \wedge x = y))$	$\forall\exists$
3. some S all P are	$\exists yPy \wedge \exists x(Sx \wedge \forall y(Py \rightarrow x = y))$	$\exists\forall$
4. some S some P are	$\exists x(Sx \wedge \exists y(Py \wedge x = y))$	$\exists\exists$
5. all S all P are not	$\forall x(Sx \rightarrow \forall y(Py \rightarrow x \neq y))$	$\forall\forall\neg$
6. all S some P are not	$\neg\exists yPy \vee \forall x(Sx \rightarrow \exists y(Py \wedge x \neq y))$	$\forall\exists\neg$
7. some S all P are not	$\neg\exists xSx \vee \exists x(Sx \wedge \forall y(Py \rightarrow x \neq y))$	$\exists\forall\neg$
8. some S some P are not	$\neg\exists xSx \vee \neg\exists yPy \vee \exists x(Sx \wedge \exists y(Py \wedge x \neq y))$	$\exists\exists\neg$

- this fragment induces the partition $\Pi_{\mathsf{FOL}}(\mathcal{F}_{unusual})$:

$$\{ \quad \forall\forall, \quad \forall\exists \wedge \forall\exists\neg, \quad \exists\forall \wedge \exists\forall\neg, \quad \forall\exists\neg \wedge \exists\forall\neg \wedge \exists\exists, \quad \forall\forall\neg \quad \}$$

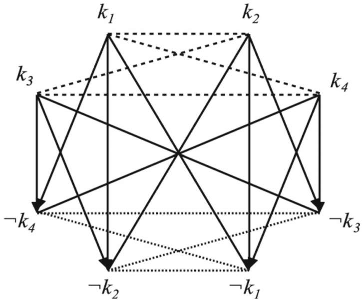
- 5 anchor formulas ⇒ bitstrings of length 5
- $\forall\forall \equiv \forall\exists \wedge \exists\forall$ $(10000 = 11000 \wedge 10100)$
- $\exists\exists \not\equiv \forall\exists \vee \exists\forall$ $(11110 \neq 11000 \vee 10100)$



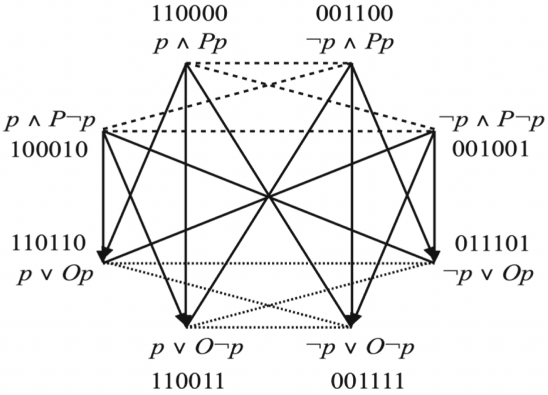
- generic description of a **Keynes-Johnson octagon**: $(\mathcal{F}_{KJ}, \mathcal{R}_{KJ})$
  - $\mathcal{F}_{KJ} = \{k_1, k_2, k_3, k_4, \neg k_1, \neg k_2, \neg k_3, \neg k_4\}$
  - $\mathcal{R}_{KJ}$: k_1 and k_3 are unconnected; k_2 and k_4 are unconnected; contrarieties between k_1 and k_2, k_1 and k_4, k_3 and k_2, k_3 and k_4

- the Keynes-Johnson octagons come in **two Boolean subtypes**:

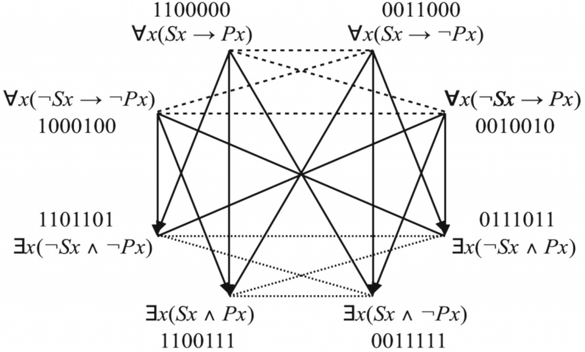
strong Keynes-Johnson octagon	$\bigvee_{i=1}^{i=4} k_i$ is a tautology	**length 6**
weak Keynes-Johnson hexagon	$\bigvee_{i=1}^{i=4} k_i$ is not a tautology	**length 7**



- induces the partition $\Pi_{\mathsf{KD}}(\mathcal{F}_{deon}) = \{ \quad p \wedge Pp \wedge P\neg p, \quad p \wedge Op, \quad \neg p \wedge Pp \wedge P\neg p, \quad \neg p \wedge Op, \quad p \wedge O\neg p, \quad \neg p \wedge O\neg p \quad \}$
- 6 anchor formulas $\Rightarrow$ bitstrings of length 6



- induces the partition $\Pi_{\text{FOL}_\exists}(\mathcal{F}_{subneg}) = \{$
  $\forall x(Sx \to Px) \wedge \forall x(\neg Sx \to \neg Px)$, $\forall x(Sx \to Px) \wedge \exists x(\neg Sx \wedge Px)$,
  $\forall x(Sx \to \neg Px) \wedge \forall x(\neg Sx \to Px)$, $\forall x(Sx \to \neg Px) \wedge \exists x(\neg Sx \wedge \neg Px)$,
  $\exists x(Sx \wedge \neg Px) \wedge \forall x(\neg Sx \to \neg Px)$, $\exists x(Sx \wedge Px) \wedge \forall x(\neg Sx \to Px)$,
  $\exists x(Sx \wedge Px) \wedge \exists x(Sx \wedge \neg Px) \wedge \exists x(\neg Sx \wedge Px) \wedge \exists x(\neg Sx \wedge \neg Px)$ $\}$
- 7 anchor formulas $\Rightarrow$ bitstrings of length 7



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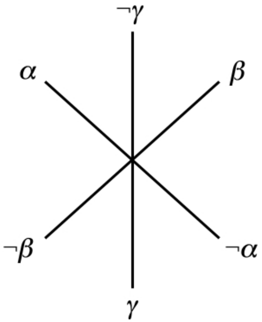
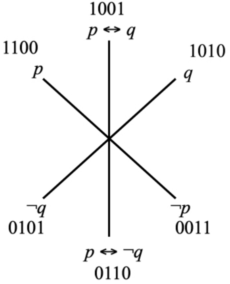
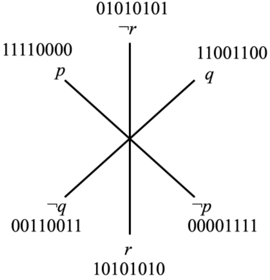
- generic description of a **U12 hexagon**: $(\mathcal{F}_{U12}, \mathcal{R}_{U12})$
  - $\mathcal{F}_{U12} = \{\alpha, \beta, \gamma, \neg\alpha, \neg\beta, \neg\gamma\}$
  - $\mathcal{R}_{U12}$: α, β, γ are pairwise unconnected

- the U12 hexagons come in **five Boolean subtypes**:
  U12 hexagons that require bitstrings of length 4, 5, 6, 7, 8



- the fragment $\mathcal{F}_\ell = \{p, q, \neg p, \neg q, p \leftrightarrow q, p \leftrightarrow \neg q\}$ induces the partition $\Pi_{\mathsf{CPL}}(\mathcal{F}_\ell) = \{p \wedge q, p \wedge \neg q, \neg p \wedge q, \neg p \wedge \neg q\}$ $\Rightarrow$ length 4
- the fragment $\mathcal{F}_r = \{p, q, r, \neg p, \neg q, \neg r\}$ induces the partition $\Pi_{\mathsf{CPL}}(\mathcal{F}_r) = \{p \wedge q \wedge r, p \wedge q \wedge \neg r, p \wedge \neg q \wedge r, p \wedge \neg q \wedge \neg r, \neg p \wedge q \wedge r, \neg p \wedge q \wedge \neg r, \neg p \wedge \neg q \wedge r, \neg p \wedge \neg q \wedge \neg r\}$ $\Rightarrow$ length 8





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Boolean homogeneity

- some Aristotelian families do not have multiple Boolean subtypes:
  - they are **Boolean homogeneous**
  - all their members can be encoded using bitstrings of the **same length**

- some examples:
  - the family of PCDs: requires only bitstrings of length 2
  - the family of classical squares: requires only bitstrings of length 3
  - the family of degenerate squares: requires only bitstrings of length 4
  - the family of SC hexagons: requires only bitstrings of length 4
  - the family of Lenzen octagons: requires only bitstrings of length 5

- note:
  - the most well-known family (classical squares) is Boolean homogeneous
  - this might explain why the issue of Boolean subtypes is not very familiar

- what determines whether a given Aristotelian family $\mathcal{A}$ has multiple Boolean subtypes or is rather Boolean homogeneous?

- given fragment $\mathcal{F} = \{\varphi_1, \ldots, \varphi_m\}$ and logic S, recall that

  $\Pi_\mathsf{S}(\mathcal{F}) := \{\alpha \in \mathcal{L} \mid \alpha = \pm\varphi_1 \wedge \cdots \wedge \pm\varphi_m$, and α is S-consistent$\}$

  (the elements $\alpha \in \Pi_\mathsf{S}(\mathcal{F})$ are **anchor formulas**) ☞ lecture 1

- how to calculate the partition induced by the **generic description** $(\mathcal{F}_\mathcal{A}, \mathcal{R}_\mathcal{A})$ of some Aristotelian family $\mathcal{A}$
  (recall: this is independent of any concrete logical system)

- $\Pi_\mathcal{A} := \{\alpha \in \mathcal{L} \mid \alpha = \pm\varphi_1 \wedge \cdots \wedge \pm\varphi_m$, and α is $\mathcal{A}$-consistent$\}$

- an anchor formula α is **$\mathcal{A}$-consistent** iff it does not contain two conjuncts that are contradictory or contrary according to $\mathcal{R}_\mathcal{A}$

- an anchor formula is
  - $\mathcal{A}$-**consistent** iff it does not contain two conjuncts that are contradictory or contrary according to the generic description of $\mathcal{A}$ (viz., $\mathcal{R}_{\mathcal{A}}$)
  - $\mathcal{A}$-**inconsistent** iff it does contain two conjuncts that are contradictory or contrary according to the generic description of $\mathcal{A}$ (viz., $\mathcal{R}_{\mathcal{A}}$)

- lemma: if an anchor formula is $\mathcal{A}$-inconsistent, then it is S-inconsistent (contrapositive: if it is S-consistent, then it is $\mathcal{A}$-consistent)

- the converse does **not** hold: an anchor formula can be S-inconsistent and yet $\mathcal{A}$-consistent

- concrete example: $(p \vee q) \wedge \neg p \wedge \neg q$
  - this formula is CPL-inconsistent
  - this formula is $\mathcal{A}$-consistent (for any Aristotelian family $\mathcal{A}$)

- we have just seen:
  - if an anchor formula is $\mathcal{A}$-inconsistent, then it is S-inconsistent
  - an anchor formula can be S-inconsistent and yet $\mathcal{A}$-consistent
  - example: $(p \vee q) \wedge \neg p \wedge \neg q$ (**three** conjuncts)

- lemma: consider an anchor formula with **at most two** conjuncts:
  - if that anchor formula is $\mathcal{A}$-inconsistent, then it is S-inconsistent
  - if that anchor formula is S-inconsistent, then it is $\mathcal{A}$-inconsistent

  $\Rightarrow$ $\mathcal{A}$-consistency guarantees S-consistency

- lemma: consider an anchor formula with **at least three** conjuncts:
  - if that anchor formula is $\mathcal{A}$-inconsistent, then it is S-inconsistent
  - that anchor formula can be S-inconsistent and yet $\mathcal{A}$-consistent

  $\Rightarrow$ $\mathcal{A}$-consistency does not guarantee S-consistency

Boolean subtypes vs. Boolean homogeneity

- what determines whether a given Aristotelian family $\mathcal{A}$ has multiple Boolean subtypes or is rather Boolean homogeneous?

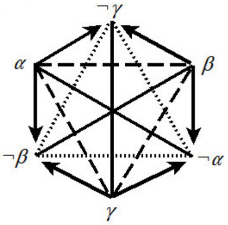
- $\Pi_{\mathcal{A}} = \{\alpha \in \mathcal{L} \mid \alpha = \pm\varphi_1 \wedge \cdots \wedge \pm\varphi_m$, and α is $\mathcal{A}$-consistent$\}$

- each anchor formula $\alpha \in \Pi_{\mathcal{A}}$ is $\mathcal{A}$-consistent
  - if α has at most two conjuncts, it is also guaranteed to be S-consistent
  - if α has at least three conjuncts, it is not guaranteed to be S-consistent

- **case distinction**:
  - all $\alpha \in \Pi_{\mathcal{A}}$ are guaranteed to be S-consistent
    $\Rightarrow \mathcal{A}$ is **Boolean homogeneous**, with single bitstring length $|\Pi_{\mathcal{A}}|$
  - $n > 0$ formulas in $\Pi_{\mathcal{A}}$ are not guaranteed to be S-consistent
    $\Rightarrow \mathcal{A}$ has $n+1$ **Boolean subtypes**, with bitstring lengths
    $|\Pi_{\mathcal{A}}| - n, \ldots, |\Pi_{\mathcal{A}}| - 1, |\Pi_{\mathcal{A}}|$

- anchor formulas in Π_{JSB}:
  - α — guaranteed to be S-consistent
  - β — guaranteed to be S-consistent
  - γ — guaranteed to be S-consistent
  - $\neg\alpha \wedge \neg\beta \wedge \neg\gamma$ — **not** guaranteed to be S-consistent

- the Aristotelian family of JSB hexagons has 2 Boolean subtypes:
  - length 3, corresponding to partition $\{\alpha, \beta, \gamma\}$
  - length 4, corresponding to partition $\{\alpha, \beta, \gamma, \neg\alpha \wedge \neg\beta \wedge \neg\gamma\}$



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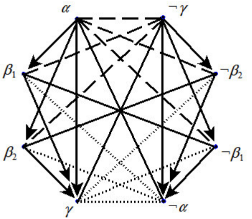
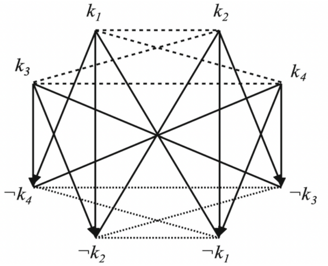
- anchor formulas in Π_{Buri}:
  - α — guaranteed to be S-consistent
  - $\neg\alpha \wedge \beta_1 \wedge \beta_2$ — **not** guaranteed to be S-consistent
  - $\beta_1 \wedge \neg\beta_2$ — guaranteed to be S-consistent
  - $\neg\beta_1 \wedge \beta_2$ — guaranteed to be S-consistent
  - $\neg\beta_1 \wedge \neg\beta_2 \wedge \gamma$ — **not** guaranteed to be S-consistent
  - $\neg\gamma$ — guaranteed to be S-consistent
- the Aristotelian family of Buridan octagons has 3 Boolean subtypes: length 6, length 5, length 4

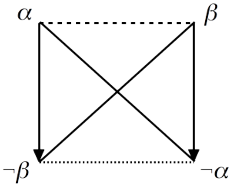


- anchor formulas in Π_{KJ}:
  - $k_1 \wedge k_3$ guaranteed to be S-consistent
  - $k_1 \wedge \neg k_3$ guaranteed to be S-consistent
  - $k_2 \wedge k_4$ guaranteed to be S-consistent
  - $k_2 \wedge \neg k_4$ guaranteed to be S-consistent
  - $\neg k_1 \wedge k_3$ guaranteed to be S-consistent
  - $\neg k_2 \wedge k_4$ guaranteed to be S-consistent
  - $\neg k_1 \wedge \neg k_2 \wedge \neg k_3 \wedge \neg k_4$ **not** guaranteed to be S-consistent
- the KJ octagons have 2 Boolean subtypes: length 7, length 6

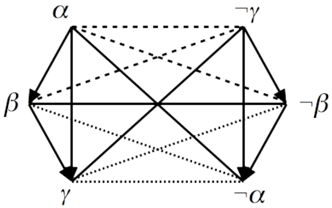


- anchor formulas in Π_{class_sq}:
  - α — guaranteed to be S-consistent
  - β — guaranteed to be S-consistent
  - $\neg\alpha \wedge \neg\beta$ — guaranteed to be S-consistent

- the Aristotelian family of classical squares is Boolean homogeneous (length 3)



- anchor formulas in Π_{SC}:
  - α guaranteed to be S-consistent
  - $\neg\alpha \wedge \beta$ guaranteed to be S-consistent
  - $\neg\beta \wedge \gamma$ guaranteed to be S-consistent
  - $\neg\gamma$ guaranteed to be S-consistent
- the Aristotelian family of SC hexagons is Boolean homogeneous (length 4)



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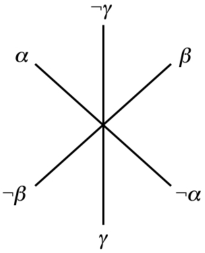
- anchor formulas in Π_{U12}:
  - $\alpha \wedge \beta \wedge \gamma$ — not guaranteed to be S-consistent
  - $\alpha \wedge \beta \wedge \neg\gamma$ — not guaranteed to be S-consistent
  - $\alpha \wedge \neg\beta \wedge \gamma$ — not guaranteed to be S-consistent
  - $\alpha \wedge \neg\beta \wedge \neg\gamma$ — not guaranteed to be S-consistent
  - $\neg\alpha \wedge \beta \wedge \gamma$ — not guaranteed to be S-consistent
  - $\neg\alpha \wedge \beta \wedge \neg\gamma$ — not guaranteed to be S-consistent
  - $\neg\alpha \wedge \neg\beta \wedge \gamma$ — not guaranteed to be S-consistent
  - $\neg\alpha \wedge \neg\beta \wedge \neg\gamma$ — not guaranteed to be S-consistent



- anchor formulas in Π_{U12}:
  - $\alpha \wedge \beta \wedge \gamma$ — not guaranteed to be S-consistent
  - $\alpha \wedge \beta \wedge \neg\gamma$ — not guaranteed to be S-consistent
  - $\alpha \wedge \neg\beta \wedge \gamma$ — not guaranteed to be S-consistent
  - $\alpha \wedge \neg\beta \wedge \neg\gamma$ — not guaranteed to be S-consistent
  - $\neg\alpha \wedge \beta \wedge \gamma$ — not guaranteed to be S-consistent
  - $\neg\alpha \wedge \beta \wedge \neg\gamma$ — not guaranteed to be S-consistent
  - $\neg\alpha \wedge \neg\beta \wedge \gamma$ — not guaranteed to be S-consistent
  - $\neg\alpha \wedge \neg\beta \wedge \neg\gamma$ — not guaranteed to be S-consistent

- misguided prediction: the Aristotelian family of U12 hexagons has 9 Boolean subtypes: length 8, 7, 6, 5, 4, 3, 2, 1, 0

- **but** encoding unconnectedness requires bitstrings of length at least 4

☞ lecture 2

- correct analysis: the Aristotelian family of U12 hexagons has 5 Boolean subtypes: length 8, 7, 6, 5, 4

- ongoing research effort in logical geometry:
  develop a **systematic typology** of Aristotelian diagrams
- for each diagram size:
  what are the **Aristotelian families** with that size?
- for each Aristotelian family:
  what are the **Boolean subfamilies** of that Aristotelian family?
- further **series** of Aristotelian families:
  - e.g. α_n-structure: n pairwise contrary formulas, and their negations
  - α_1 = PCD, α_2 = classical square, α_3 = JSB, α_4 = Moretti ...

	σ0	σ1	σ2		σ3					σ4																					
	empty	PCD	classical	degenerate	JSB	SC	U4	U12	U8	I	III = Moretti	II = Beziau	IV = Buridan	V	VI = Lenzen	VII	IX	VIII	X	XIV	XV	XIII	XI	XII	XVII = KeyJon	XVIII	XVI	1	2	3	4
length 1	X																														
length 2		X																													
length 3			X		X																										
length 4				X	X	X	X	X		X	X	X	X	X																	
length 5							X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X							
length 6								X	X			X	X	X		X	X	X	X	X	X	X	X	X	X	X	X				
length 7								X						X				X	X	X	X	X	X	X	X	X	X				
length 8								X						X						X	X	X	X	X		X	X				
length 9														X						X	X	X	X	X			X				
length 10																						X	X	X							
length 11																							X	X							
length 12																							X	X							
length 13																								X							
length 14																								X							
length 15																								X							
length 16																								X							
length 17																															

- recall from lecture 1:
  if $\mathcal{F}$ only contains S-contingent formulas and is closed under negation, then $\lceil \log_2(|\mathcal{F}|+2) \rceil \quad \leq \quad |\Pi_S(\mathcal{F})| \quad \leq \quad 2^{|\mathcal{F}|/2}$
- some specific cases:
  - $|\mathcal{F}| = 2 \quad \Rightarrow \quad 2 = \lceil \log_2(2+2) \rceil \quad \leq \quad |\Pi_S(\mathcal{F})| \quad \leq \quad 2^{2/2} = 2$
  - $|\mathcal{F}| = 4 \quad \Rightarrow \quad 3 = \lceil \log_2(4+2) \rceil \quad \leq \quad |\Pi_S(\mathcal{F})| \quad \leq \quad 2^{4/2} = 4$
  - $|\mathcal{F}| = 6 \quad \Rightarrow \quad 3 = \lceil \log_2(6+2) \rceil \quad \leq \quad |\Pi_S(\mathcal{F})| \quad \leq \quad 2^{6/2} = 8$
  - $|\mathcal{F}| = 8 \quad \Rightarrow \quad 4 = \lceil \log_2(8+2) \rceil \quad \leq \quad |\Pi_S(\mathcal{F})| \quad \leq \quad 2^{8/2} = 16$

	σ_0	σ_1	σ_2		σ_3					σ_4																					
	empty	PCD	classical	degenerate	JSB	SC	U4	U12	U8	I	III = Moretti	II = Beziau	IV = Buridan	V	VI = Lenzen	VII	IX	VIII	X	XIV	XV	XIII	XI	XII	XVII = KeyJon	XVIII	XVI	1	2	3	4
length 1	X																														
length 2		X																													
length 3			X		X																										
length 4				X	X	X	X	X		X	X	X	X	X																	
length 5							X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X							
length 6								X	X			X	X	X		X	X	X	X	X	X	X	X	X	X	X	X				
length 7								X						X				X	X	X	X	X	X	X	X	X	X				
length 8								X						X						X	X	X	X	X		X	X				
length 9														X						X	X	X	X	X			X				
length 10																						X	X	X							
length 11																							X	X							
length 12																							X	X							
length 13																								X							
length 14																								X							
length 15																								X							
length 16																								X							

- recall from lecture 1:
  if $\mathcal{F}$ only contains S-contingent formulas and is closed under negation,
  then $2\lceil \log_2(|\Pi_S(\mathcal{F})|)\rceil \quad \leq \quad |\mathcal{F}| \quad \leq \quad 2^{|\Pi_S(\mathcal{F})|} - 2$
- some specific cases:
  - $|\Pi_S(\mathcal{F})| = 2 \quad \Rightarrow \quad 2 = 2\lceil \log_2(2)\rceil \quad \leq \quad |\mathcal{F}| \quad \leq \quad 2^2 - 2 = 2$
  - $|\Pi_S(\mathcal{F})| = 3 \quad \Rightarrow \quad 4 = 2\lceil \log_2(3)\rceil \quad \leq \quad |\mathcal{F}| \quad \leq \quad 2^3 - 2 = 6$
  - $|\Pi_S(\mathcal{F})| = 4 \quad \Rightarrow \quad 4 = 2\lceil \log_2(4)\rceil \quad \leq \quad |\mathcal{F}| \quad \leq \quad 2^4 - 2 = 14$
  - $|\Pi_S(\mathcal{F})| = 5 \quad \Rightarrow \quad 6 = 2\lceil \log_2(5)\rceil \quad \leq \quad |\mathcal{F}| \quad \leq \quad 2^5 - 2 = 30$

	σ_0	σ_1	σ_2		σ_3					σ_4																					
	empty	PCD	classical	degenerate	JSB	SC	U4	U12	U8	I	III = Moretti	II = Beziau	IV = Buridan	V	VI = Lenzen	VII	IX	VIII	X	XIV	XV	XIII	XI	XII	XVII = Keylon	XVIII	XVI	1	2	3	4
length 1	X																														
length 2		X																													
length 3			X		X																										
length 4				X	X	X	X	X		X	X	X	X	X																	
length 5							X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X							
length 6								X	X			X	X	X		X	X	X	X	X	X	X	X	X	X	X	X				
length 7								X						X				X	X	X	X	X	X	X	X	X	X				
length 8								X						X						X	X	X	X	X		X	X				
length 9														X						X	X	X	X	X			X				
length 10																						X	X	X							
length 11																							X	X							
length 12																							X	X							
length 13																								X							
length 14																								X							
length 15																								X							
length 16																								X							

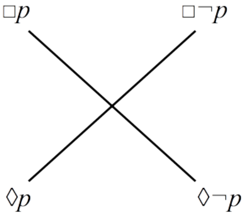
1. Basic Concepts and Bitstring Semantics
2. Abstract-Logical Properties of Aristotelian Diagrams, Part I
   ☞ Aristotelian, Opposition, Implication and Duality Relations
3. Visual-Geometric Properties of Aristotelian Diagrams
   ☞ Informational Equivalence, Symmetry and Distance
4. **Abstract-Logical Properties of Aristotelian Diagrams, Part II**
   ☞ Boolean Structure and **Logic-Sensitivity**
5. Case Studies and Philosophical Outlook

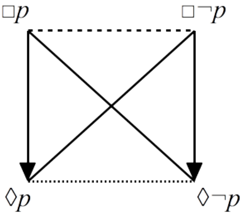
- Aristotelian diagrams are (in various ways) **sensitive** to the specific details of the underlying logical system
  - informal definition: 'φ and ψ cannot be true together'
  - model-theoretic definition: $\models_{\mathsf{S}} \neg(\varphi \wedge \psi)$

- this point has also been emphasized by Claudio Pizzi:
  - "An obvious but frequently neglected proviso concerning the squares of oppositions is that the relations which are claimed to hold between the formulas of the square only subsist with reference to some given background system" (2016)
  - "an ordered 4-[tu]ple is an Aristotelian square always with respect to some system S" (2017)

- one fragment $\mathcal{F} := \{\Box p, \Diamond p, \Box\neg p, \Diamond\neg p\}$
- two logical systems:
  - K: basic normal modal logic (all Kripke models)
  - KD: axiom $\Box p \rightarrow \Diamond p$ (or $\Diamond\top$) (serial Kripke models)

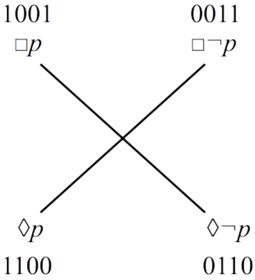
- $(\mathcal{F}, \mathsf{K})$ is a **degenerate square**
- $(\mathcal{F}, \mathsf{KD})$ is a **classical square**

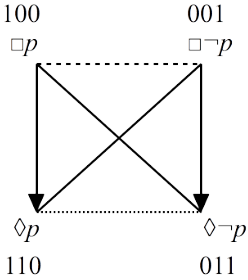




Basic example: bitstring analysis

- $\Pi_{\mathsf{K}}(\mathcal{F}) = \{\Box p \wedge \Diamond p, \Diamond p \wedge \Diamond\neg p, \Box\neg p \wedge \Diamond\neg p, \Box p \wedge \Box\neg p\}$ (length 4)
- $\Pi_{\mathsf{KD}}(\mathcal{F}) = \{\Box p, \Diamond p \wedge \Diamond\neg p, \Box\neg p\}$ (length 3)
- from K to KD: **delete the fourth bit position**
  - $\Box p \wedge \Box\neg p$ is K-consistent, but KD-inconsistent
  - $\Pi_{\mathsf{KD}}(\mathcal{F}) = \{\alpha \in \Pi_{\mathsf{K}}(\mathcal{F}) \mid \alpha$ is KD-consistent$\}$





- Aristotelian relations are logic-sensitive:
  - $\Box p$ and $\Box\neg p$ are K-unconnected, but KD-contrary
  - $\Box p$ and $\Diamond p$ are K-unconnected, but in KD-subalternation

- duality relations are not (or rather: less) logic-sensitive:
  - $\Box p$ and $\Box\neg p$ are each other's internal negation, in K as well as KD
  - $\Box p$ and $\Diamond p$ are each other's dual, in K as well as KD

- duality relations **are** sensitive to the underlying logic, but only to **non-Boolean** aspects
  - K and KD are both classical Boolean logics $\Rightarrow$ no differences in duality
  - e.g. $p \wedge q$ and $p \vee q$ are dual in classical logic, but not in intuitionistic logic

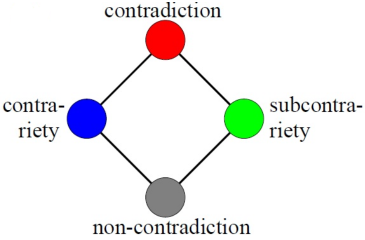
- recall from lecture 2:
  - $\mathcal{AG}_\mathsf{S}$ is hybrid between $\mathcal{OG}_\mathsf{S}$ and $\mathcal{IG}_\mathsf{S}$
  - non-contradiction: $NCD_\mathsf{S}(\varphi, \psi)$ iff $\not\models_\mathsf{S} \neg(\varphi \wedge \psi)$ and $\not\models_\mathsf{S} \neg(\neg\varphi \wedge \neg\psi)$

- consider two logics W,S, and assume that for all φ: $\models_\mathsf{W} \varphi \quad \Rightarrow \quad \models_\mathsf{S} \varphi$

- theorem: from the weaker logic to the stronger logic
  - if $CD_\mathsf{W}(\varphi, \psi)$ then $CD_\mathsf{S}(\varphi, \psi)$
  - if $C_\mathsf{W}(\varphi, \psi)$ then $C_\mathsf{S}(\varphi, \psi)$ or $CD_\mathsf{S}(\varphi, \psi)$
  - if $SC_\mathsf{W}(\varphi, \psi)$ then $SC_\mathsf{S}(\varphi, \psi)$ or $CD_\mathsf{S}(\varphi, \psi)$
  - if $NCD_\mathsf{W}(\varphi, \psi)$ then $NCD_\mathsf{S}(\varphi, \psi)$ or $C_\mathsf{S}(\varphi, \psi)$ or $SC_\mathsf{S}(\varphi, \psi)$ or $CD_\mathsf{S}(\varphi, \psi)$

- theorem: from the stronger logic to the weaker logic
  - if $CD_\mathsf{S}(\varphi, \psi)$ then $CD_\mathsf{W}(\varphi, \psi)$ or $C_\mathsf{W}(\varphi, \psi)$ or $SC_\mathsf{W}(\varphi, \psi)$ or $NCD_\mathsf{W}(\varphi, \psi)$
  - if $C_\mathsf{S}(\varphi, \psi)$ then $C_\mathsf{W}(\varphi, \psi)$ or $NCD_\mathsf{W}(\varphi, \psi)$
  - if $SC_\mathsf{S}(\varphi, \psi)$ then $SC_\mathsf{W}(\varphi, \psi)$ or $NCD_\mathsf{W}(\varphi, \psi)$
  - if $NCD_\mathsf{S}(\varphi, \psi)$ then $NCD_\mathsf{W}(\varphi, \psi)$

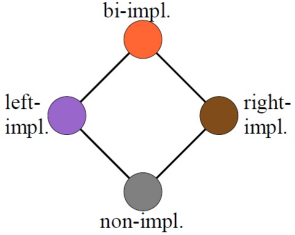
- diagrammatic summary of the two theorems:
  - going to a stronger logic can make you go up in the diagram
  - going to a weaker logic can make you go down in the diagram



- recall the **informativity ordering** $\leq_i^\forall$ of $\mathcal{OG}$ ☞ lecture 2
- theorem: the following are equivalent:
  - S is at least as strong as W
  - for all φ, ψ: if $R_\mathsf{W}(\varphi, \psi)$ and $R'_\mathsf{S}(\varphi, \psi)$, then $R \leq_i^\forall R'$

Logic-sensitivity of the implication relations

- a completely analogous story can be told about the implication relations
- summary:
  - going to a stronger logic can make you go up in the diagram
  - going to a weaker logic can make you go down in the diagram



- sources of logic-sensitivity in Aristotelian diagrams:
  - logic-sensitivity of the Aristotelian (opp./imp.) **relations** themselves
  - the condition that Aristotelian diagrams only contain **pairwise non-equivalent** formulas
  - the condition that Aristotelian diagrams only contain **contingent** formulas

- **equivalence** is a logic-sensitive notion:
  two formulas might be equivalent in one logic,
  and not equivalent in another logic

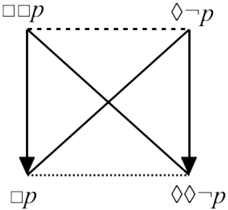
- **contingency** is a logic-sensitive notion:
  a formula might be contingent in one logic, and not in another logic

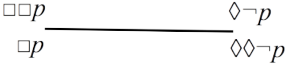
The non-equivalence condition: example

- one fragment $\mathcal{F} := \{\Box\Box p, \Box p, \Diamond\Diamond\neg p, \Diamond\neg p\}$
- two logical systems:
  - KT: axiom $\Box p \to p$ (reflexive Kripke models)
  - KT4: axioms $\Box p \to p$, $\Box p \to \Box\Box p$ (refl., transitive Kripke models)

- $(\mathcal{F}, \mathsf{KT})$ is a **classical square**
- $(\mathcal{F}, \mathsf{KT4})$ is a **PCD**

- we go from $C_{\mathsf{KT}}(\Box\Box p, \Diamond\neg p)$ to $CD_{\mathsf{KT4}}(\Box\Box p, \Diamond\neg p)$
- we go from $LI_{\mathsf{KT}}(\Box\Box p, \Box p)$ to $BI_{\mathsf{KT4}}(\Box\Box p, \Box p)$





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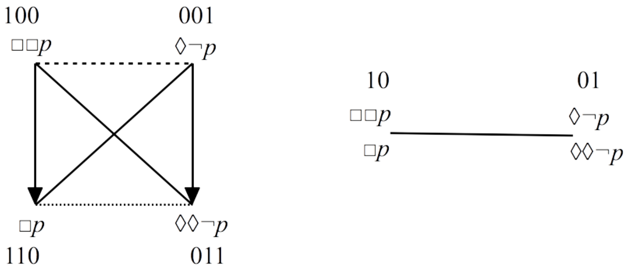
- $\Pi_{\mathsf{KT}}(\mathcal{F}) = \{\Box\Box p, \Box p \wedge \Diamond\Diamond\neg p, \Diamond\neg p\}$ (length 3)
- $\Pi_{\mathsf{KT4}}(\mathcal{F}) = \{\Box p, \Diamond\neg p\}$ (length 2)

- from KT to KT4: **delete the second bit position**
  - $\Box p \wedge \Diamond\Diamond\neg p$ is KT-consistent, but KT4-inconsistent
  - $\Pi_{\mathsf{KT4}}(\mathcal{F}) = \{\alpha \in \Pi_{\mathsf{KT}}(\mathcal{F}) \mid \alpha$ is KT4-consistent$\}$



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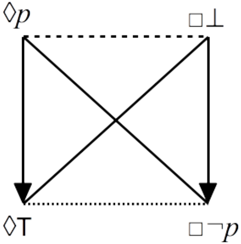
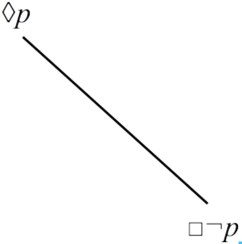
- one fragment $\mathcal{F} := \{\Diamond p, \Diamond\top, \Box\neg p, \Box\bot\}$
- two logical systems:
  - K: basic normal modal logic (all Kripke models)
  - KD: axiom $\Box p \to \Diamond p$ (or $\Diamond\top$) (serial Kripke models)

- $(\mathcal{F}, \mathsf{K})$ is a **classical square**
- $(\mathcal{F}, \mathsf{KD})$ is a **PCD**

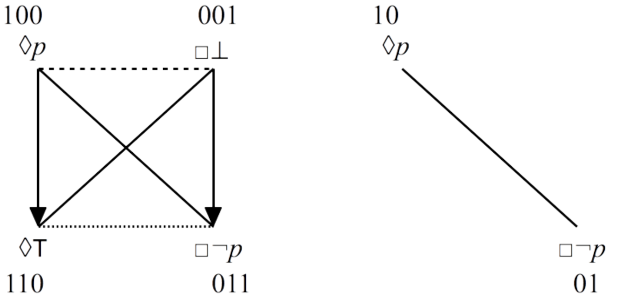
- $\Diamond\top$ is contingent in K, but a tautology in KD
- $\Box\bot$ is contingent in K, but a contradiction in KD





- $\Pi_{\mathsf{K}}(\mathcal{F}) = \{\Diamond p, \Diamond\top \wedge \Box\neg p, \Box\bot\}$ (length 3)
- $\Pi_{\mathsf{KD}}(\mathcal{F}) = \{\Diamond p, \Box\neg p\}$ (length 2)

- from K to KD: **delete the third bit position**
  - $\Box\bot$ is K-consistent, but KD-inconsistent
  - $\Pi_{\mathsf{KD}}(\mathcal{F}) = \{\alpha \in \Pi_{\mathsf{K}}(\mathcal{F}) \mid \alpha$ is KD-consistent$\}$



1. Basic Concepts and Bitstring Semantics

2. Abstract-Logical Properties of Aristotelian Diagrams, Part I
   ☞ Aristotelian, Opposition, Implication and Duality Relations

3. Visual-Geometric Properties of Aristotelian Diagrams
   ☞ Informational Equivalence, Symmetry and Distance

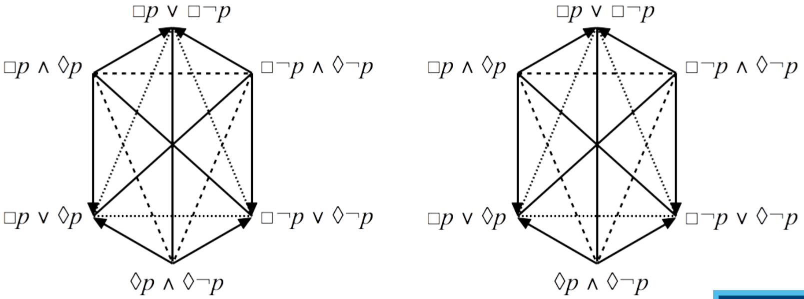
4. **Abstract-Logical Properties of Aristotelian Diagrams, Part II**
   ☞ **Boolean Structure and Logic-Sensitivity**

5. Case Studies and Philosophical Outlook

KU LEUVEN

- logic-sensitivity: one fragment, two logics $\Rightarrow$ two different diagrams
  - classical square vs. degenerate square
  - classical square vs. PCD
  - JSB hexagon vs. classical square
  - Buridan octagon vs. Lenzen octagon
- until now: the two diagrams belong to **different Aristotelian families**
- also possible:
  - the two diagrams belong to the **same Aristotelian family**
  - but to **different Boolean subtypes** of that Aristotelian family

- one fragment $\mathcal{F}$, two logical systems: K and KD
- $\mathcal{F} := \{\Box p \wedge \Diamond p, \Box p \vee \Diamond p, \Box\neg p \wedge \Diamond\neg p, \Box\neg p \vee \Diamond\neg p, \Box p \vee \Box\neg p, \Diamond p \wedge \Diamond\neg p\}$
- $(\mathcal{F}, \mathsf{K})$ is a **weak JSB hexagon**
- $(\mathcal{F}, \mathsf{KD})$ is a **strong JSB hexagon**
- $\not\models_{\mathsf{K}} (\Box p \wedge \Diamond p) \vee (\Box\neg p \wedge \Diamond\neg p) \vee (\Diamond p \wedge \Diamond\neg p)$
- $\models_{\mathsf{KD}} (\Box p \wedge \Diamond p) \vee (\Box\neg p \wedge \Diamond\neg p) \vee (\Diamond p \wedge \Diamond\neg p)$



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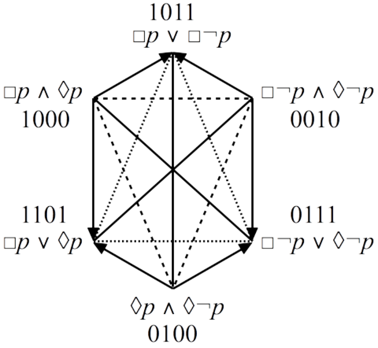
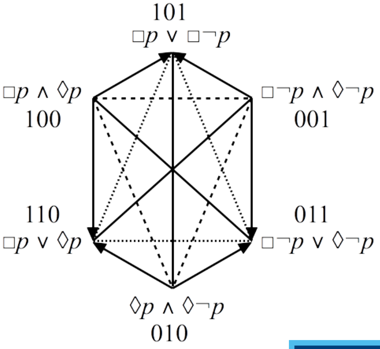
- $\Pi_{\mathsf{K}}(\mathcal{F}) = \{\Box p \wedge \Diamond p, \Diamond p \wedge \Diamond\neg p, \Box\neg p \wedge \Diamond\neg p, \varphi_{long}\}$ (length 4)
- $\Pi_{\mathsf{KD}}(\mathcal{F}) = \{\Box p, \Diamond p \wedge \Diamond\neg p, \Box\neg p\}$ (length 3)
- $\varphi_{long} := (\Box p \vee \Diamond p) \wedge (\Box\neg p \vee \Diamond\neg p) \wedge (\Box p \vee \Box\neg p)$
- from K to KD: **delete the fourth bit position**
  (φ_{long} is K-consistent, but KD-inconsistent)





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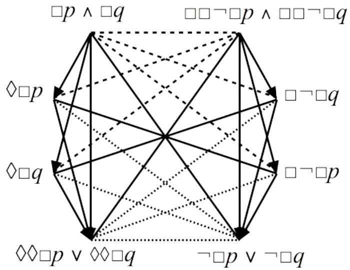
- one fragment $\mathcal{F}$, three logics: KT, KT4 (= S4) and KT45 (= S5)
- $\mathcal{F} := \{\Box p \wedge \Box q, \Diamond\Box p, \Diamond\Box q, \Diamond\Diamond\Box p \vee \Diamond\Diamond\Box q\}$ (+ negations)
- $(\mathcal{F}, \text{KT})$ is a **weak Buridan octagon**
- $(\mathcal{F}, \text{KT4})$ is an **intermediate Buridan octagon**
- $(\mathcal{F}, \text{KT45})$ is a **strong Buridan octagon**
- $\Box p \wedge \Box q \not\equiv_{\text{KT}} \Diamond\Box p \wedge \Diamond\Box q$ and $\Diamond\Diamond\Box p \vee \Diamond\Diamond\Box q \not\equiv_{\text{KT}} \Diamond\Box p \vee \Diamond\Box q$
- $\Box p \wedge \Box q \not\equiv_{\text{KT4}} \Diamond\Box p \wedge \Diamond\Box q$ and $\Diamond\Diamond\Box p \vee \Diamond\Diamond\Box q \equiv_{\text{KT4}} \Diamond\Box p \vee \Diamond\Box q$
- $\Box p \wedge \Box q \equiv_{\text{KT45}} \Diamond\Box p \wedge \Diamond\Box q$ and $\Diamond\Diamond\Box p \vee \Diamond\Diamond\Box q \equiv_{\text{KT45}} \Diamond\Box p \vee \Diamond\Box q$



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- many different types of logic-sensitivity:
  - based on the Aristotelian relations
  - based on the diagrammatic condition of non-equivalence
  - based on the diagrammatic condition of contingency
  - based on Boolean subtypes

- there are many **cross-connections** among these different types

- example:
  - for any 4-formula fragment $\mathcal{F} = \{\varphi, \psi, \neg\varphi, \neg\psi\}$, define a 6-formula fragment $H(\mathcal{F}) := \{\varphi \wedge \psi, \varphi \vee \psi, \neg\varphi \wedge \neg\psi, \neg\varphi \vee \neg\psi, \varphi \vee \neg\psi, \neg\varphi \wedge \psi\}$
  - theorem: for any logical system S:
    - if $(\mathcal{F}, \mathsf{S})$ is a degenerate square, then $(H(\mathcal{F}), \mathsf{S})$ is a weak JSB hexagon
    - if $(\mathcal{F}, \mathsf{S})$ is a classical square, then $(H(\mathcal{F}), \mathsf{S})$ is a strong JSB hexagon

Thank you!

Questions?

More info: `www.logicalgeometry.org`

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