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Introduction to Logical Geometry

2. Abstract-Logical Properties of Aristotelian Diagrams, Part I

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1. Basic Concepts, Bitstring Semantics and (Iso)morphisms
2. **Abstract-Logical Properties of Aristotelian Diagrams, Part I**
   ☞ **Aristotelian, Opposition, Implication and Duality Relations**
3. Visual-Geometric Properties of Aristotelian Diagrams
   ☞ Informational Equivalence, Cognition, Symmetry and Distance
4. Abstract-Logical Properties of Aristotelian Diagrams, Part II
   ☞ Boolean Structure and Logic-Sensitivity
5. Case Studies and Philosophical Outlook

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5. Case Studies and Philosophical Outlook

- recall the **Aristotelian geometry** $\mathcal{AG}_\mathsf{S} = \{CD_\mathsf{S}, C_\mathsf{S}, SC_\mathsf{S}, SA_\mathsf{S}\}$ (relative to an appropriate logical system S)

- φ and ψ are said to be

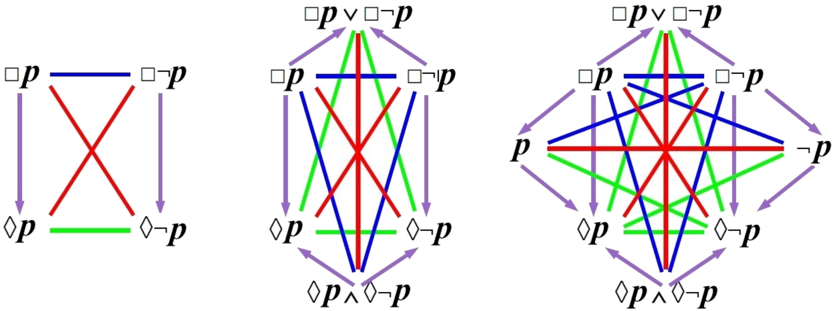
S-contradictory (CD_S)	iff	$\models_\mathsf{S} \neg(\varphi \wedge \psi)$	and	$\models_\mathsf{S} \neg(\neg\varphi \wedge \neg\psi)$
S-contrary (C_S)	iff	$\models_\mathsf{S} \neg(\varphi \wedge \psi)$	and	$\not\models_\mathsf{S} \neg(\neg\varphi \wedge \neg\psi)$
S-subcontrary (SC_S)	iff	$\not\models_\mathsf{S} \neg(\varphi \wedge \psi)$	and	$\models_\mathsf{S} \neg(\neg\varphi \wedge \neg\psi)$
in S-subalternation (SA_S)	iff	$\models_\mathsf{S} \varphi \rightarrow \psi$	and	$\not\models_\mathsf{S} \psi \rightarrow \varphi$

- **Aristotelian square of opposition**: 4 propositions + the Aristotelian relations holding between them

Generalizations of the Aristotelian square

- throughout history: several proposals to extend the square of opposition
  - more propositions, more relations
  - larger and more complex diagrams
  - hexagons, octagons, cubes and other three-dimensional figures

- cf. the motivating examples from lecture 1



The success of the Aristotelian square

- the square and its extensions: various types of hexagons, octagons, etc.
- the extensions are very interesting
  - well-motivated (propositional logic, modal logic S5)
  - throughout history (William of Sherwood, John Buridan, John N. Keynes)
  - interrelations (e.g. JSB hexagon is Boolean closure of classical square)

- yet there is a stunning discrepancy:
  - (nearly) **all** logicians know about the Aristotelian square of opposition
  - (nearly) **no** logicians know about the other Aristotelian diagrams

- our explanation: "the Aristotelian square is very **informative**"
  - this claim sounds intuitive, but is also vague
  - we will provide a precise and well-motivated framework

- recall the Aristotelian geometry $\mathcal{AG}_{\mathsf{S}}$: φ and ψ are said to be

S-contradictory (CD_{S})	iff	$\models_{\mathsf{S}} \neg(\varphi \wedge \psi)$	and	$\models_{\mathsf{S}} \neg(\neg\varphi \wedge \neg\psi)$
S-contrary (C_{S})	iff	$\models_{\mathsf{S}} \neg(\varphi \wedge \psi)$	and	$\not\models_{\mathsf{S}} \neg(\neg\varphi \wedge \neg\psi)$
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in S-subalternation (SA_{S})	iff	$\models_{\mathsf{S}} \varphi \rightarrow \psi$	and	$\not\models_{\mathsf{S}} \psi \rightarrow \varphi$

- problems with the relations of $\mathcal{AG}_{\mathsf{S}}$:
  - **not mutually exclusive**: e.g. $\bot$ and p are contrary and subaltern in CPL (lemma: if φ, ψ are contingent, they stand in at most one Arist. relation)
  - **not exhaustive**: e.g. p and $\Diamond p \wedge \Diamond\neg p$ are in no Arist. relation at all in S5 (lemma: if φ is contingent, then φ stands in no Arist. relation to itself)
  - **conceptual confusion**: can be true/false together vs. truth propagation
    - ‘together’ $\rightsquigarrow$ symmetrical relations (undirected)
    - ‘propagation’ $\rightsquigarrow$ asymmetrical relations (directed)

- replace subalternation with 'non-contradiction'
- two formulas φ and ψ are said to be

S-contradictory (CD_S)	iff	$\models_S \neg(\varphi \wedge \psi)$	and	$\models_S \neg(\neg\varphi \wedge \neg\psi)$
S-contrary (C_S)	iff	$\models_S \neg(\varphi \wedge \psi)$	and	$\not\models_S \neg(\neg\varphi \wedge \neg\psi)$
S-subcontrary (SC_S)	iff	$\not\models_S \neg(\varphi \wedge \psi)$	and	$\models_S \neg(\neg\varphi \wedge \neg\psi)$
S-non-contradictory (NCD_S)	iff	$\not\models_S \neg(\varphi \wedge \psi)$	and	$\not\models_S \neg(\neg\varphi \wedge \neg\psi)$

- the **opposition geometry** for S: $\mathcal{OG}_S := \{CD_S, C_S, SC_S, NCD_S\}$
- Carnapian state descriptions ('rows 1 and 4 of a truth table'):
  - $\Sigma_1(\varphi, \psi) := \varphi \wedge \psi$
  - $\Sigma_4(\varphi, \psi) := \neg\varphi \wedge \neg\psi$

  (note: 'symmetry' between conjuncts of Σ_1 and Σ_4)
- $\mathcal{OG}_S$ is defined of terms $\neg\Sigma_1$ and $\neg\Sigma_4$

- subalternation: truth propagation 'from left to right' $\rightsquigarrow$ left-implication
- vary the 'direction' of truth propagation

- two formulas φ and ψ are said to be in

S-bi-implication (BI_S)	iff	$\models_S \varphi \to \psi$	and	$\models_S \psi \to \varphi$
S-left-implication (LI_S)	iff	$\models_S \varphi \to \psi$	and	$\not\models_S \psi \to \varphi$
S-right-implication (RI_S)	iff	$\not\models_S \varphi \to \psi$	and	$\models_S \psi \to \varphi$
S-non-implication (NI_S)	iff	$\not\models_S \varphi \to \psi$	and	$\not\models_S \psi \to \varphi$

- the **implication geometry** for S: $\mathcal{IG}_S := \{BI_S, LI_S, RI_S, NI_S\}$

- Carnapian state descriptions ('rows 2 and 3 of a truth table'):
  - $\Sigma_2(\varphi, \psi) := \varphi \wedge \neg\psi$
  - $\Sigma_3(\varphi, \psi) := \neg\varphi \wedge \psi$

(note: 'asymmetry' between conjuncts of Σ_2 and Σ_3)

- $\mathcal{IG}_S$ is defined of terms $\neg\Sigma_2$ and $\neg\Sigma_3$

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- two new geometries: opposition geometry and implication geometry
- together, they solve the problems of the Aristotelian geometry

- the relations of $\mathcal{OG}_{\mathsf{S}}$ are mutually exclusive and jointly exhaustive: each pair of formulas stands in exactly one opposition relation

- the relations of $\mathcal{IG}_{\mathsf{S}}$ are mutually exclusive and jointly exhaustive: each pair of formulas stands in exactly one implication relation

- no longer conceptual confusion:
  - $\mathcal{OG}_{\mathsf{S}}$ is uniformly defined in terms of being able to be true/false together (cf. the symmetrical state descriptions Σ_1 and Σ_4)
  - $\mathcal{IG}_{\mathsf{S}}$ is uniformly defined in terms of truth propagation (cf. the asymmetrical state descriptions Σ_2 and Σ_3)

- clear link with Correia (2012):
  two distinct philosophical traditions in interpreting the square:
  - square as a theory of negation — commentaries on *De Interpretatione*
  - square as a theory of consequence — commentaries on *Prior Analytics*
- terminological remark:
  - 'square of opposition', 'hexagon of opposition', 'cube of opposition'
  - misnomer: exclusive focus on $\mathcal{OG}_{\mathsf{S}}$, while ignoring $\mathcal{IG}_{\mathsf{S}}$
  - more appropriate terminology: 'Aristotelian square' etc.
  - concrete examples from the literature:
    - 'square of opposition and equipollence' (John Mikhail, 2007)
    - 'square of implication and opposition' (W. E. Johnson, 1922)
    - 'octagon of implication and opposition' (W. E. Johnson, 1922)

- opposition and implication geometry are conceptually independent yet there's a clear relationship between them (symmetry breaking):

$$\begin{array}{lcl} CD_{\mathsf{S}}(\varphi,\psi) & \Leftrightarrow & BI_{\mathsf{S}}(\varphi,\neg\psi) \\ C_{\mathsf{S}}(\varphi,\psi) & \Leftrightarrow & LI_{\mathsf{S}}(\varphi,\neg\psi) \\ SC_{\mathsf{S}}(\varphi,\psi) & \Leftrightarrow & RI_{\mathsf{S}}(\varphi,\neg\psi) \\ NCD_{\mathsf{S}}(\varphi,\psi) & \Leftrightarrow & NI_{\mathsf{S}}(\varphi,\neg\psi) \end{array}$$

- both geometries are also internally structured:

$$\begin{array}{lcl} CD_{\mathsf{S}}(\varphi,\psi) & \Leftrightarrow & CD_{\mathsf{S}}(\neg\varphi,\neg\psi) \\ C_{\mathsf{S}}(\varphi,\psi) & \Leftrightarrow & SC_{\mathsf{S}}(\neg\varphi,\neg\psi) \\ SC_{\mathsf{S}}(\varphi,\psi) & \Leftrightarrow & C_{\mathsf{S}}(\neg\varphi,\neg\psi) \\ NCD_{\mathsf{S}}(\varphi,\psi) & \Leftrightarrow & NCD_{\mathsf{S}}(\neg\varphi,\neg\psi) \end{array} \qquad \begin{array}{lcl} BI_{\mathsf{S}}(\varphi,\psi) & \Leftrightarrow & BI_{\mathsf{S}}(\neg\varphi,\neg\psi) \\ LI_{\mathsf{S}}(\varphi,\psi) & \Leftrightarrow & RI_{\mathsf{S}}(\neg\varphi,\neg\psi) \\ RI_{\mathsf{S}}(\varphi,\psi) & \Leftrightarrow & LI_{\mathsf{S}}(\neg\varphi,\neg\psi) \\ NI_{\mathsf{S}}(\varphi,\psi) & \Leftrightarrow & NI_{\mathsf{S}}(\neg\varphi,\neg\psi) \end{array}$$

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- given φ, ψ, we define a binary, truth-functional connective $\circ^{(\varphi,\psi)} = (\circ_1, \circ_2, \circ_3, \circ_4) \in \{0,1\}^4$:
  - φ, ψ stand in exactly one opposition relation
    for $i = 1, 4$, define $\circ_i := \begin{cases} 0 & \text{if } \models_{\mathsf{S}} \neg\Sigma_i(\varphi,\psi) \\ 1 & \text{if } \not\models_{\mathsf{S}} \neg\Sigma_i(\varphi,\psi) \end{cases}$
  - φ, ψ stand in exactly one implication relation
    for $i = 2, 3$, define $\circ_i := \begin{cases} 0 & \text{if } \models_{\mathsf{S}} \neg\Sigma_i(\varphi,\psi) \\ 1 & \text{if } \not\models_{\mathsf{S}} \neg\Sigma_i(\varphi,\psi) \end{cases}$

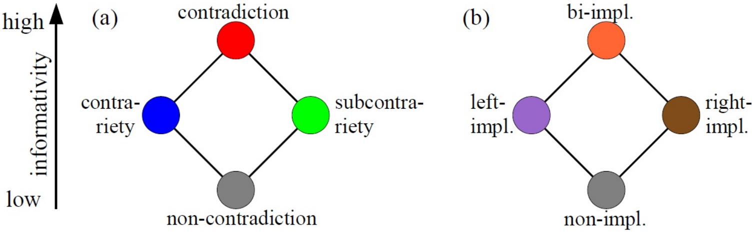
- **theorem**: for all φ, ψ, it holds that $\models \varphi \circ^{(\varphi,\psi)} \psi$
  - e.g.: if $SC_{\mathsf{S}}(\varphi,\psi)$ and $NI_{\mathsf{S}}(\varphi,\psi)$, then $\circ^{(\varphi,\psi)} = (1,1,1,0)$, so $\models_{\mathsf{S}} \varphi \vee \psi$
  - e.g.: if $C_{\mathsf{S}}(\varphi,\psi)$ and $RI_{\mathsf{S}}(\varphi,\psi)$, then $\circ^{(\varphi,\psi)} = (0,1,0,1)$, so $\models_{\mathsf{S}} \neg\psi$

- **theorem**: if φ and ψ are contingent, they can stand in only 7 of the possible 16 $(= 4 \times 4)$ combinations of an opp. and an imp. relation

- general idea: the informativity of a statement σ is inversely correlated with the size of its information range $\mathbb{I}(\sigma)$
- **informativity ordering** $\leq_i$: $\sigma \leq_i \tau$ iff $\mathbb{I}(\sigma) \supseteq \mathbb{I}(\tau)$
- we are interested in statements of the form $R_{\mathsf{S}}(\varphi, \psi)$, with $R_{\mathsf{S}} \in \mathcal{OG}_{\mathsf{S}} \cup \mathcal{IG}_{\mathsf{S}}$
- $\mathbb{I}(R_{\mathsf{S}}(\varphi, \psi)) := \{\mathbb{M} \in \mathcal{C}_{\mathsf{S}} \mid \mathbb{M} \text{ is compatible with } R_{\mathsf{S}}(\varphi, \psi)\}$
- a model $\mathbb{M}$ of the logic S is said to be **compatible** with $R_{\mathsf{S}}(\varphi, \psi)$ iff for all $1 \leq i \leq 4 : \Big(R_{\mathsf{S}}(\varphi, \psi) \;\Rightarrow \models_{\mathsf{S}} \neg\Sigma_i(\varphi, \psi)\Big) \Longrightarrow \mathbb{M} \models \neg\Sigma_i(\varphi, \psi)$
- lift informativity ordering from **statements** $R_{\mathsf{S}}(\varphi, \psi)$ to **relations** R_{S}: $R_{\mathsf{S}} \leq_i^{\forall} S_{\mathsf{S}}$ iff $\forall \varphi, \psi : R_{\mathsf{S}}(\varphi, \psi) \leq_i S_{\mathsf{S}}(\varphi, \psi)$

- for $1 \leq i \leq 4$, models of type i are those that make $\Sigma_i(\varphi, \psi)$ true
- informativity of the opposition and implication relations:

	models of type		models of type
$CD_S(\varphi, \psi)$	2,3	$BI_S(\varphi, \psi)$	1, 4
$C_S(\varphi, \psi)$	2,3,4	$LI_S(\varphi, \psi)$	1, 3,4
$SC_S(\varphi, \psi)$	1,2,3	$RI_S(\varphi, \psi)$	1,2, 4
$NCD_S(\varphi, \psi)$	1,2,3,4	$NI_S(\varphi, \psi)$	1,2,3,4



- close match between formal account and intuitions:
  - e.g. CD_{S} is more informative than C_{S}
  - if φ is known,
    - announcing $CD_{\mathsf{S}}(\varphi, \psi)$ uniquely determines ψ
    - announcing $C_{\mathsf{S}}(\varphi, \psi)$ does not uniquely determine ψ

- combinatorial results on finite Boolean algebras ($\sim$ bitstrings!)
  - Boolean algebra $\mathbb{B}$ with 2^n formulas, formula of level i:
    - 1 contradictory
    - $2^{n-i} - 1$ contraries and $2^i - 1$ subcontraries
    - $(2^{n-i} - 1)(2^i - 1)$ non-contradictories
  - $1 < 2^{n-i} - 1, 2^i - 1 < (2^{n-i} - 1)(2^i - 1)$ iff $1 < i < n - 1$

- coherence with earlier results:
  - $\mathcal{OG}_{\mathsf{S}}$ and $\mathcal{IG}_{\mathsf{S}}$ yield isomorphic informativity lattices
  - $CD_{\mathsf{S}}(\varphi, \psi) \Leftrightarrow BI_{\mathsf{S}}(\varphi, \neg\psi)$ etc.

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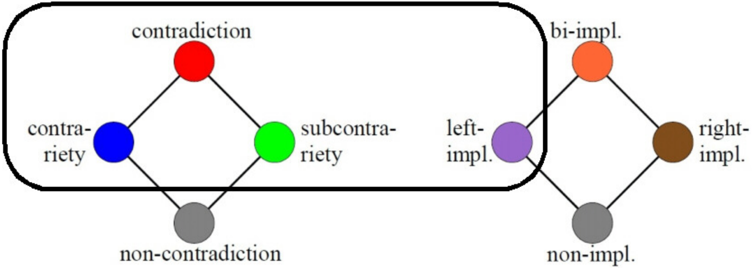
The special status of the Aristotelian square

- why is the Aristotelian square special?

- our answer: because it is very informative
  - it is a very informative **diagram** (viz. no unconnectedness)
  - in a very informative **geometry** (viz. the Aristotelian geometry)

- Aristotelian geometry: hybrid between
  - opposition geometry: contradiction, contrariety, subcontrariety
  - implication geometry: left-implication (subalternation)

- these relations are highly informative (in their geometries)



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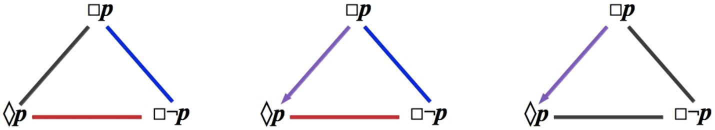
- given any two formulas:
  - they stand in exactly one opposition relation R
  - they stand in exactly one implication relation S

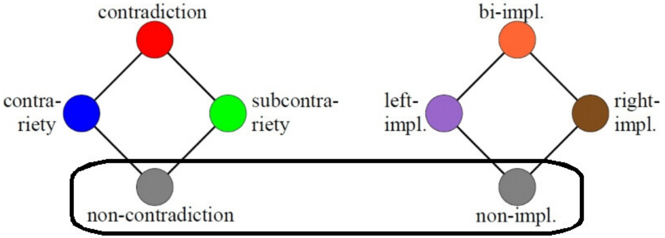
- **theorem**:
  - if R is strictly more informative than S, then R is Aristotelian
  - if S is strictly more informative than R, then S is Aristotelian

- three examples (in S5):
  - $\Box p$ and $\Diamond p$: non-contradiction and **left-implication**
  - $\Box p$ and $\Box\neg p$: **contrariety** and non-implication
  - $\Diamond p$ and $\Box\neg p$: **contradiction** and non-implication



- given any two formulas: one opposition relation, one implication relation
- what if **neither** relation is strictly more informative than the other?

- **theorem**: this can only occur in one case: NCD + NI (unconnectedness)



- Aristotelian gap = information gap
  - no Aristotelian relation at all (recall that $\mathcal{AG}_S$ is not exhaustive)
  - combination of the two least informative relations

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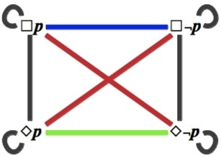
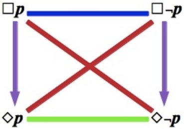
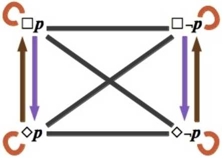
- recall the four-condition characterization of unconnectedness:
  - φ and ψ can be true together — cf. $\Sigma_1(\varphi,\psi)$
  - φ can be true while ψ is false — cf. $\Sigma_2(\varphi,\psi)$
  - φ can be false while ψ is true — cf. $\Sigma_3(\varphi,\psi)$
  - φ and ψ can be false together — cf. $\Sigma_4(\varphi,\psi)$
- unconnectedness as the combination of non-contradiction (Σ_1, Σ_4) and non-implication (Σ_2, Σ_3)
- encoding unconnectedness requires **bitstrings of length at least 4**
  - if $\mathbb{B}_\mathsf{S}(\mathcal{F}) \cong \{0,1\}^n$ for $n < 4$, then $\mathcal{F}$ does not contain any pair of S-unconnected formulas
  - if $\mathcal{F}$ contains at least one pair of S-unconnected formulas, then $\mathbb{B}_\mathsf{S}(\mathcal{F}) \cong \{0,1\}^n$ for $n \geq 4$

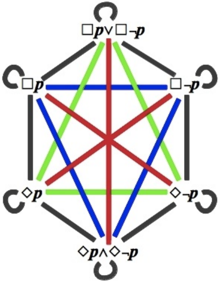
- no unconnectedness in the classical Aristotelian square

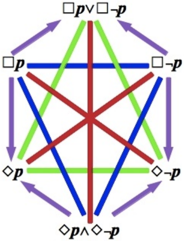


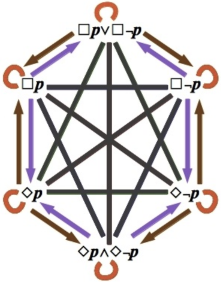




- no unconnectedness in the Jacoby-Sesmat-Blanché hexagon

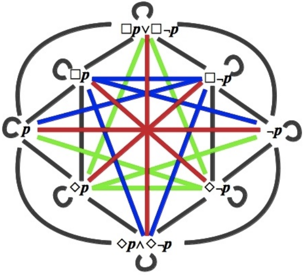


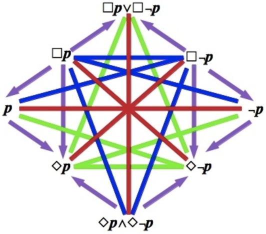


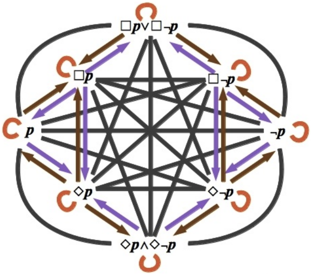


Unconnectedness in some Aristotelian diagrams, II

- unconnectedness in the Béziau octagon
- e.g. p and $\Diamond p \wedge \Diamond\neg p$ are unconnected







- the Aristotelian **geometry** is hybrid between opposition and implication
- in order to **maximize informativity**

⇒ applies to all Aristotelian diagrams

- on the level of individual **diagrams**: avoid unconnectedness
- in order to **minimize uninformativity**

⇒ some Aristotelian diagrams succeed better than others
  - classical square, JSB hexagon, SC hexagon don't have unconnectedness
  - Béziau octagon (and many other diagrams) do have unconnectedness

- Q: what about, say, the JSB hexagon and SC hexagon?
(equally informative as the square, yet less widely known)
- A: this requires yet another geometry: **duality**

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- **square of opposition**:
  - visually represents the **Aristotelian relations** of contradiction, contrariety, subcontrariety and subalternation
  - nearly always also exhibits another type of logical relations, viz. the **duality relations** of internal negation, external negation and duality
- based on the concrete examples found in literature, the notions of **Aristotelian square** and **duality square** seem almost co-extensional
- but: clear conceptual differences between the two!
- the logical and visual properties of Aristotelian and duality diagrams in isolation are relatively well-understood

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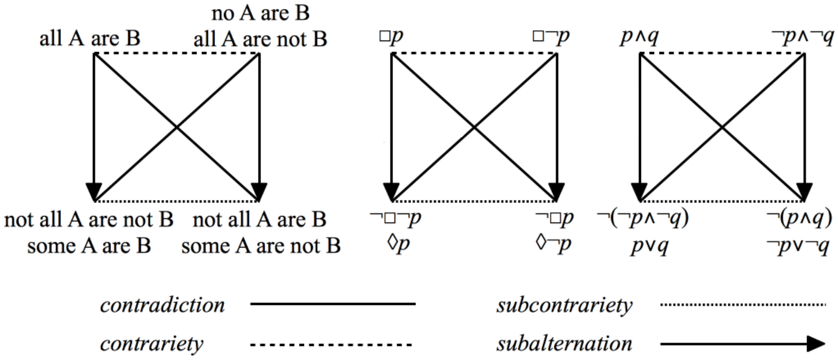
Aims and claims of this part of the lecture:

- get clearer picture of **interconnections** between the two types of relations
- introduce a new type of diagram to visualise these interconnections: the **Aristotelian/Duality Multigraph** (ADM)
- **octagons** are natural extensions/generalizations of the classical square
    - from an Aristotelian perspective **and**
    - from a duality perspective
- the **correspondence** between Aristotelian and duality relations:
    - is **lost** on the level of individual relations and diagrams
    - is **maintained** on a more abstract level

some standard examples:



- the **contradiction relation**:
  - most important and informative Aristotelian relation: each proposition φ has a unique contradictory (up to logical equivalence), viz. $\neg\varphi$
  - almost all Aristotelian diagrams in the literature are closed under contradiction: if the diagram contains φ, then it also contains $\neg\varphi$
    $\Rightarrow$ visualized by means of central symmetry ☞ lecture 3
  - the propositions in an Aristotelian diagram can naturally be grouped into **pairs of contradictory propositions** (PCDs)

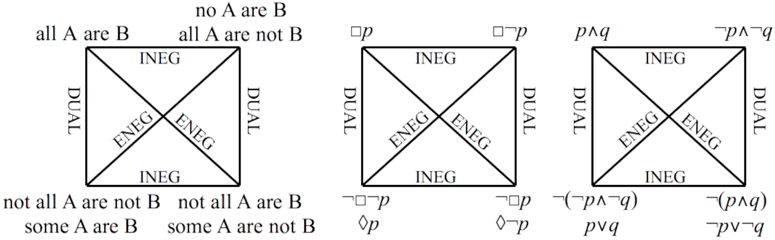
- **Aristotelian diagrams**:
  - remember the shift of perspective:
    - a square does not really consist of 4 individual propositions
    - rather, a square consists of 2 PCDs
  - natural way of extending the square: **adding more PCDs**:
    - logically: from 2 PCDs to 3 PCDs to 4 PCDs to ...
    - geometrically: from square to hexagon to octagon to ...

- suppose that two formulas φ and ψ are the results of applying n-ary operators O_φ and O_ψ to the same n propositions $\alpha_1, \ldots, \alpha_n$
- $\varphi \equiv O_\varphi(\alpha_1, \ldots, \alpha_n)$ and $\psi \equiv O_\psi(\alpha_1, \ldots, \alpha_n)$.

- φ and ψ are said to be each other's

external negation (ENEG)	iff	$O_\varphi(\alpha_1, \ldots, \alpha_n) \equiv \neg O_\psi(\alpha_1, \ldots, \alpha_n)$
internal negation (INEG)	iff	$O_\varphi(\alpha_1, \ldots, \alpha_n) \equiv O_\psi(\neg\alpha_1, \ldots, \neg\alpha_n)$
dual (DUAL)	iff	$O_\varphi(\alpha_1, \ldots, \alpha_n) \equiv \neg O_\psi(\neg\alpha_1, \ldots, \neg\alpha_n)$

the same standard examples:



- the relations are **functional** (up to logical equivalence):
  - e.g. if $\textsc{ineg}(\varphi, \psi_1)$ and $\textsc{ineg}(\varphi, \psi_2)$, then $\psi_1 \equiv \psi_2$
  - we write $\psi = \textsc{ineg}(\varphi)$ instead of $\textsc{ineg}(\varphi, \psi)$
- the relations are **symmetrical**: e.g. $\textsc{dual}(\varphi, \psi)$ iff $\textsc{dual}(\psi, \varphi)$
- the functions are **idempotent**: e.g. $\textsc{eneg}(\textsc{eneg}(\varphi)) = \varphi = \textsc{id}(\varphi)$

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- define the **identity function** $\text{ID}(\varphi) := \varphi$
- the four duality functions ID, ENEG, INEG and DUAL form a **Klein 4-group** under composition ($\circ$), with the following Cayley table:

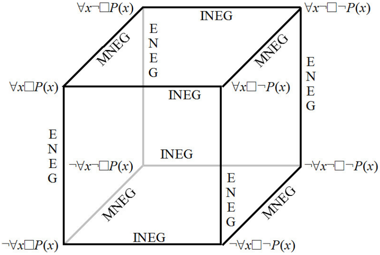
$\circ$	ID	ENEG	INEG	DUAL
ID	ID	ENEG	INEG	DUAL
ENEG	ENEG	ID	DUAL	INEG
INEG	INEG	DUAL	ID	ENEG
DUAL	DUAL	INEG	ENEG	ID

- the Klein 4-group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$:
  - each copy of $\mathbb{Z}_2$ governs its own negation
  - ID $\sim (0,0)$, ENEG $\sim (1,0)$, INEG $\sim (0,1)$, and DUAL $\sim (1,1)$

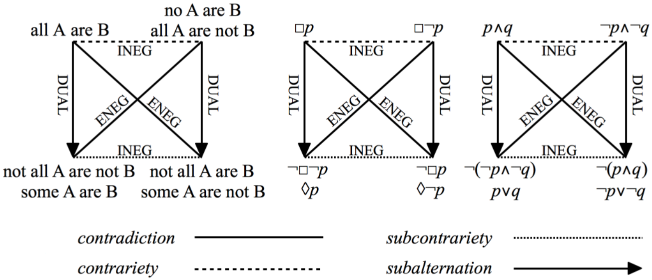
$\circ$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(0,0)$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(1,0)$	$(1,0)$	$(0,0)$	$(1,1)$	$(0,1)$
$(0,1)$	$(0,1)$	$(1,1)$	$(0,0)$	$(1,0)$
$(1,1)$	$(1,1)$	$(0,1)$	$(1,0)$	$(0,0)$

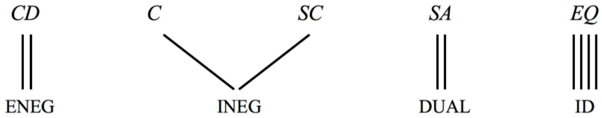
Natural way of extending the square:

- **adding more independent negation positions**
- i.e. adding more copies of $\mathbb{Z}_2$
- logically: from $\mathbb{Z}_2 \times \mathbb{Z}_2$ to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
  from 2 negation positions to 3 negation positions
  from $2^2 = 4$ duality functions to $2^3 = 8$ duality functions
- geometrically: from square to cube/octagon to ...

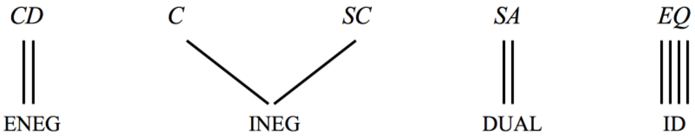


Aristotelian/duality multigraph (ADM): visualizes how many times a specific combination of Aristotelian and duality relation occurs in the square

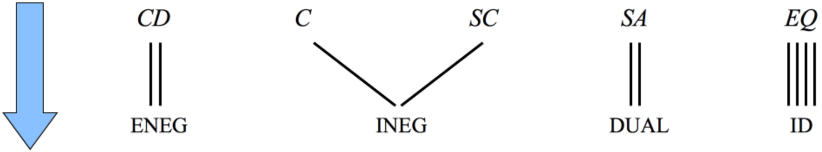




Aristotelian/Duality Multigraph (ADM)

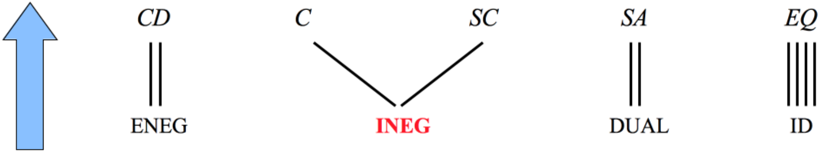


The correspondence between Aristotelian and duality relations is not perfect, but still highly regular



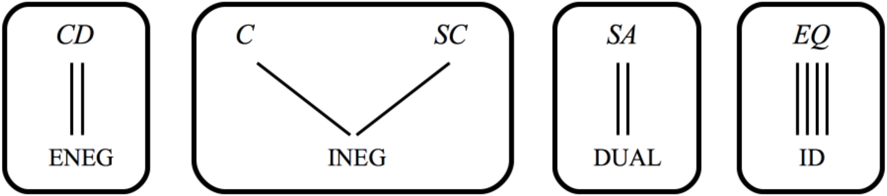
The correspondence between Aristotelian and duality relations is not perfect, but still highly regular

- each Aristotelian relation corresponds to a unique duality relation



The correspondence between Aristotelian and duality relations is not perfect, but still highly regular

- each Aristotelian relation corresponds to a unique duality relation
- vice versa, duality relations
  - ENEG, DUAL and ID correspond to a unique Aristotelian relation
  - INEG corresponds to two Aristotelian relations



The correspondence between Aristotelian and duality relations is not perfect, but still highly regular

- each Aristotelian relation corresponds to a unique duality relation
- vice versa, duality relations:
  - ENEG, DUAL and ID correspond to a unique Aristotelian relation
  - INEG corresponds to two Aristotelian relations
- ADM for the square of opposition has 4 **connected components**, viz. $\{CD, \text{ENEG}\}$, $\{C, SC, \text{INEG}\}$, $\{SA, \text{DUAL}\}$ and $\{EQ, \text{ID}\}$

- this close correspondence leads to a **quasi-identification** of the two types of squares:
  - using Aristotelian terminology to describe duality square (or vice versa)
  - viewing one as a generalization of the other
  - already noted in medieval logic (Peter of Spain, William of Sherwood):
    - mnemonic rhyme: *pre contradic, post contra, pre postque subalter*
    - ENEG = *pre* ≈ *CD*, INEG = *post* ≈ *C*, DUAL = *pre postque* ≈ *SA*

- still some **crucial differences**:
  - duality relations are all **symmetric** ⇔ Aristotelian *SA* is asymmetric
  - duality relations are all **functional** ⇔ Aristotelian *C*, *SC* and *SA* are not

    ☞ Löbner (1990, 2011), Peters & Westerståhl (2006), Westerståhl (2012)
  - duality relations are **not logic-sensitive** ⇔ Aristotelian relations are

    ☞ lecture 4

KU LEUVEN

- the most powerful way to argue for the independence of Aristotelian and duality diagrams consists in analyzing diagrams **beyond** the square
- the **hexagon** is not the most natural extension of the square:
  - natural extension from Aristotelian perspective (6 is a multiple of 2)
  - not natural extension from duality perspective (6 is not a power of 2)

  $\Rightarrow$ JSB and SC hexagon are **less informative** than classical square
- **octagon** $=$ natural extension from Aristotelian $+$ duality perspective:

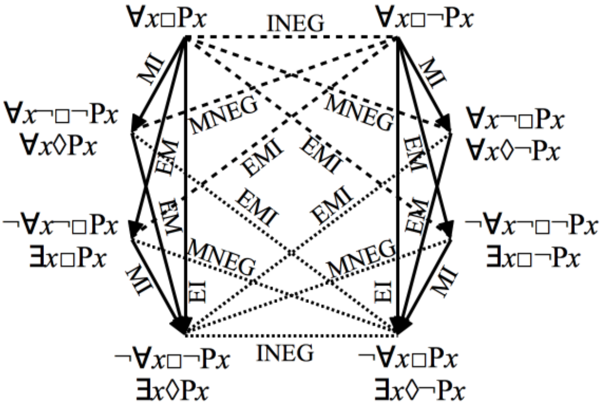
from	square	to	octagon		
	$2 \times 2 = 4 = 2^2$		$4 \times 2 = 8 = 2^3$		
	2 PCDs $\leftrightsquigarrow 2 \times 2$		4 PCDs $\leftrightsquigarrow 4 \times 2$	$\Rightarrow$	Aristotelian view
	2 negations $\leftrightsquigarrow 2^2$		3 negations $\leftrightsquigarrow 2^3$	$\Rightarrow$	duality view

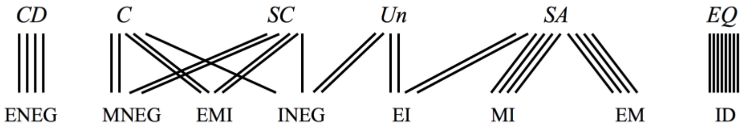
- discuss some octagons in detail:
  - **three** different **Aristotelian families** of octagons
  - **two** different types of **generalized duality**

- suppose that φ is the result of applying an n-ary composed operator $O_1 \circ O_2$ to n propositions $\alpha_1, \ldots, \alpha_n$
- $\varphi \equiv (O_1 \circ O_2)(\alpha_1, \ldots, \alpha_n) = O_1(O_2(\alpha_1, \ldots, \alpha_n))$
- an extra negation position has become available!

- the proposition $O_1(O_2(\alpha_1, \ldots, \alpha_n))$ has a unique
  - **external negation** (ENEG): $\neg O_1(\ O_2(\ \alpha_1, \ldots,\ \alpha_n))$
  - **intermediate negation** (MNEG): $O_1(\neg O_2(\ \alpha_1, \ldots,\ \alpha_n))$
  - **internal negation** (INEG): $O_1(\ O_2(\neg\alpha_1, \ldots, \neg\alpha_n))$

- 3 independent negation positions $\Rightarrow 2^3 = 8$ duality functions in total
- much richer duality behavior:
  - ENEG, MNEG, and INEG
  - ENEG $\circ$ MNEG (EM), ENEG $\circ$ INEG (EI), and MNEG $\circ$ INEG (MI)
  - ENEG $\circ$ MNEG $\circ$ INEG (EMI)


∀x□Px
INEG
∀x□¬Px
MI
MI
∀x¬□¬Px
∀x◊Px
MNEG
MNEG
∀x¬□Px
∀x◊¬Px
EM
EMI
EMI
EM
EM
EMI
EMI
EM
¬∀x¬□Px
∃x□Px
MNEG
MNEG
¬∀x¬□¬Px
∃x□¬Px
MI
EI
EI
MI
¬∀x□¬Px
∃x◊Px
INEG
¬∀x□Px
∃x◊¬Px



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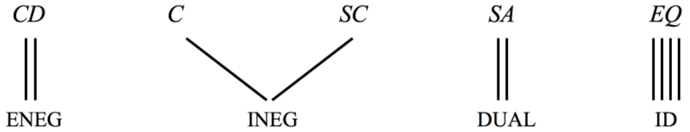

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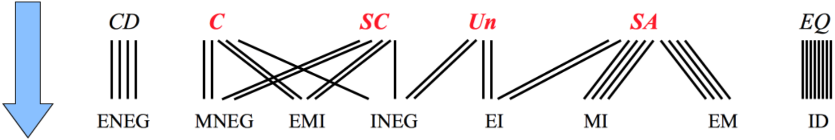

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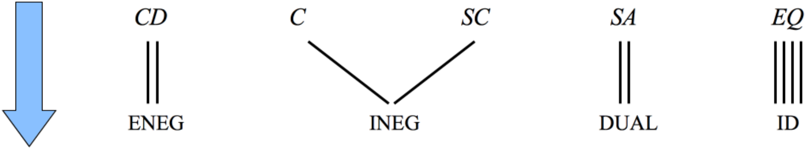

CD
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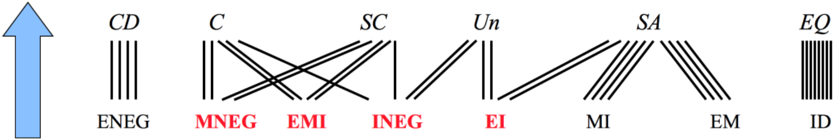
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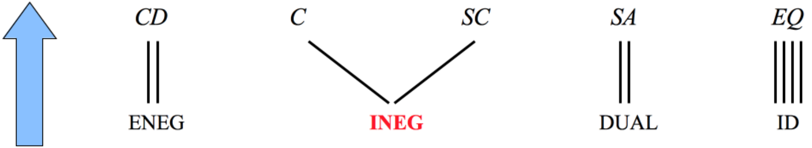
CD
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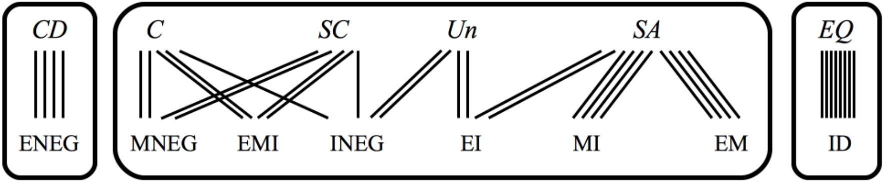
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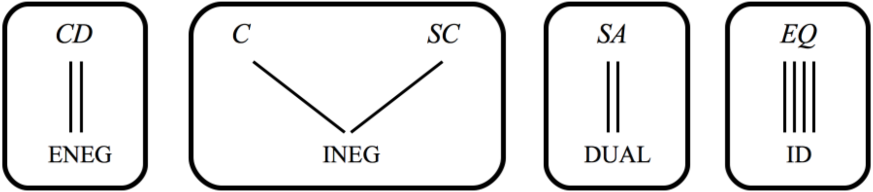
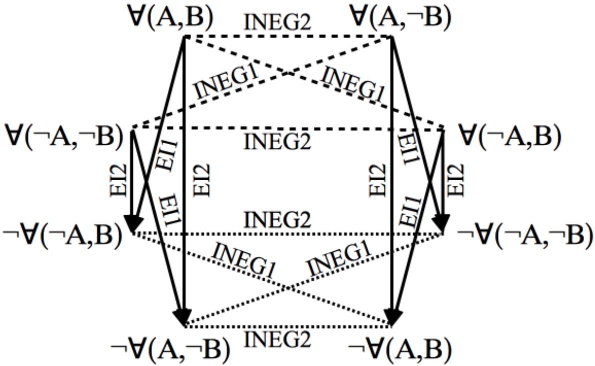
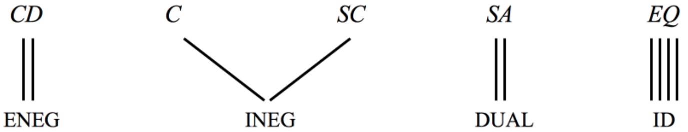
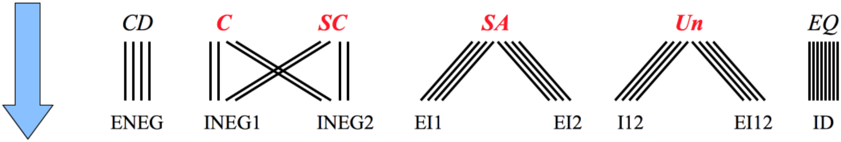

CD
ENEG
C
SC
INEG
SA
DUAL
EQ
ID

- classical duality applies internal negation to **all** arguments, i.e. the internal negation of $O(\alpha_1, \ldots, \alpha_n)$ is $O(\neg\alpha_1, \ldots, \neg\alpha_n)$
- now: apply internal negation to each argument **independently**
- with a binary operator O, we thus have 3 independent negation positions in total: the proposition $O(\alpha_1, \alpha_2)$ has a unique:
  - **external negation** (ENEG): $\neg O(\alpha_1, \alpha_2)$
  - **first internal negation** (INEG1): $O(\neg\alpha_1, \alpha_2)$,
  - **second internal negation** (INEG2): $O(\alpha_1, \neg\alpha_2)$
- 3 independent negation positions $\Rightarrow 2^3 = 8$ duality functions in total
- much richer duality behavior:
  - ENEG, INEG1, and INEG2
  - ENEG $\circ$ INEG1 (EI1), ENEG $\circ$ INEG2 (EI2), and INEG1 $\circ$ INEG2 (I12)
  - ENEG $\circ$ INEG1 $\circ$ INEG2 (EI12)


∀(A,B)
INEG2
∀(A,¬B)
INEG1
INEG1
∀(¬A,¬B)
INEG2
∀(¬A,B)
EI1
EI2
EI2
EI1
EI2
EI2
EI1
EI1
¬∀(¬A,B)
INEG2
¬∀(¬A,¬B)
INEG1
INEG1
¬∀(A,¬B)
INEG2
¬∀(A,B)

Keynes-Johnson octagon (syllogistics with subject negation)




CD
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ENEG
INEG1
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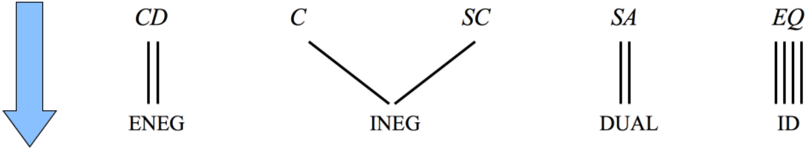
EI1
EI2
I12
EI12
ID


CD
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DUAL
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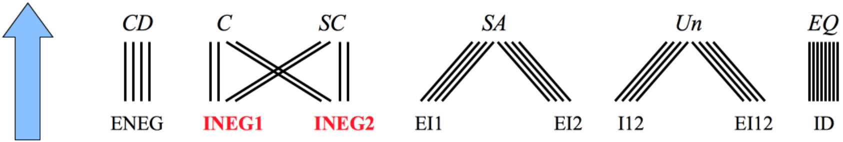
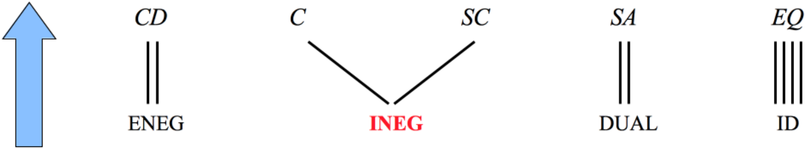
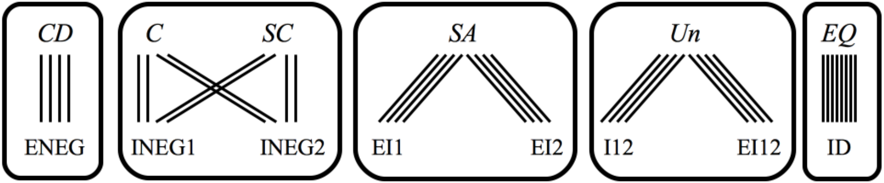
Keynes-Johnson octagon (syllogistics with subject negation)






CD
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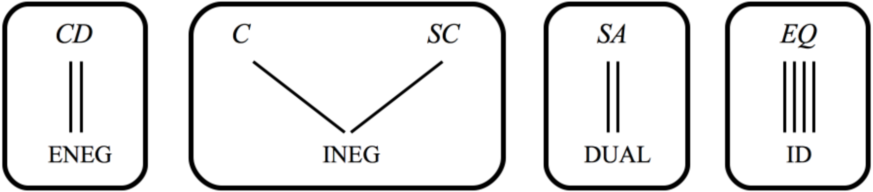
CD
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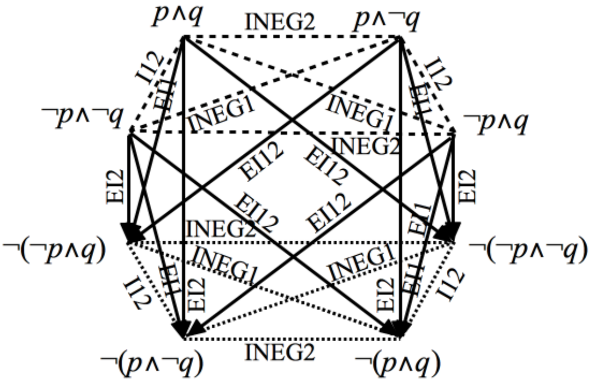
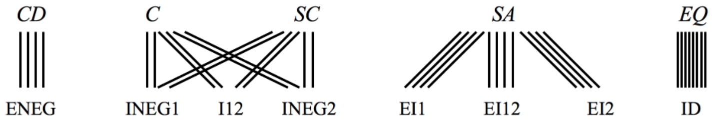
Moretti octagon (propositional logic)





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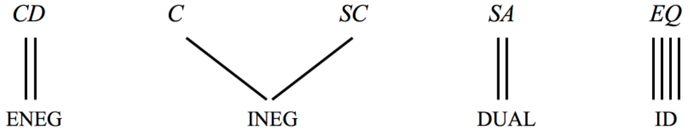

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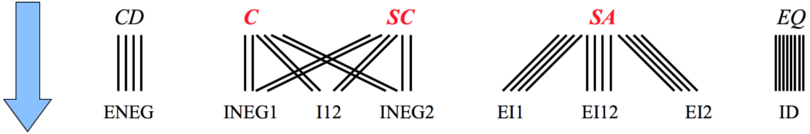
CD
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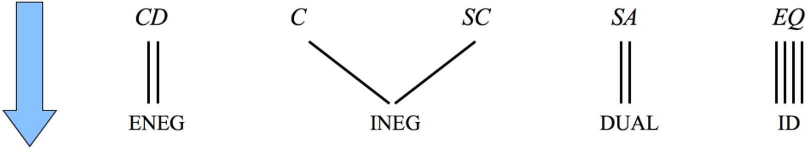
CD
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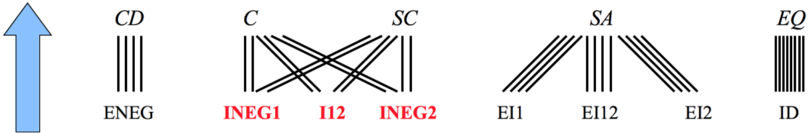
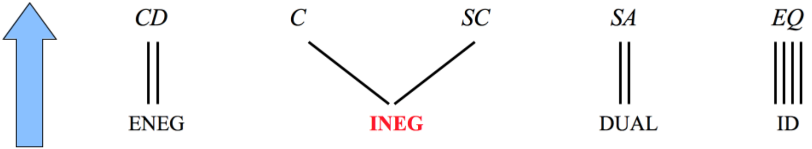
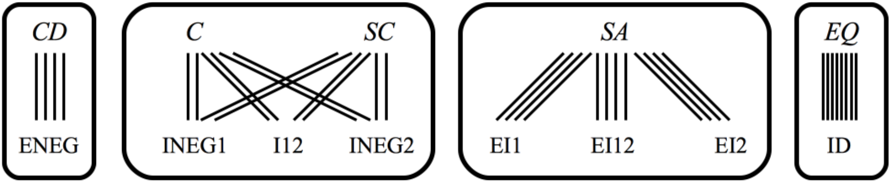
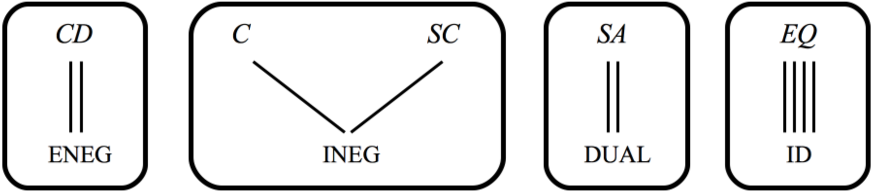
Moretti octagon (propositional logic)



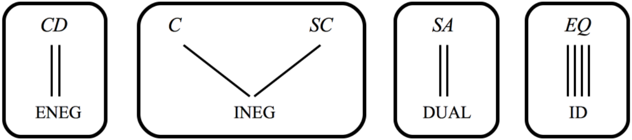


Moretti octagon (propositional logic)

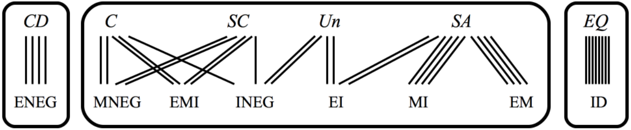




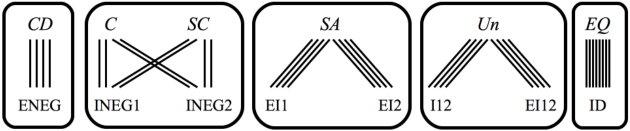
square of opposition
⇝ classical duality



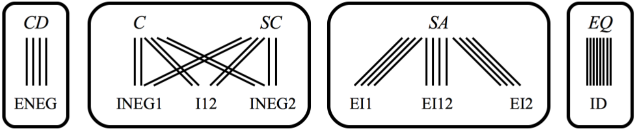
Buridan octagon
⇝ composed operator duality



Keynes-Johnson octagon
⇝ generalized Post duality



Moretti octagon
⇝ generalized Post duality



1. Basic Concepts, Bitstring Semantics and (Iso)morphisms

2. **Abstract-Logical Properties of Aristotelian Diagrams, Part I**
   ☞ Aristotelian, **Opposition, Implication and Duality Relations**

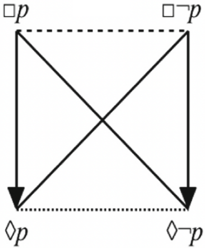
3. Visual-Geometric Properties of Aristotelian Diagrams
   ☞ Informational Equivalence, Cognition, Symmetry and Distance

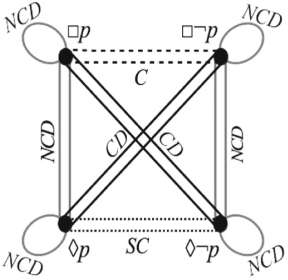
4. Abstract-Logical Properties of Aristotelian Diagrams, Part II
   ☞ Boolean Structure and Logic-Sensitivity

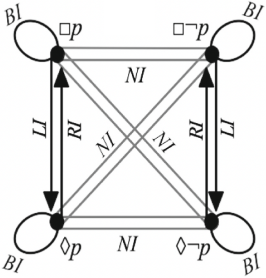
5. Case Studies and Philosophical Outlook

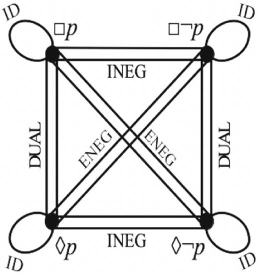
- four geometries (sets of relations):
  - Aristotelian geometry ($\mathcal{AG}$)
  - opposition geometry ($\mathcal{OG}$)
  - implication geometry ($\mathcal{IG}$)
  - duality geometry ($\mathcal{DG}$)
- Aristotelian/duality multigraph (ADM): interface between $\mathcal{AG}$ and $\mathcal{DG}$
- can't we do something similar for...
  - the interface between $\mathcal{OG}$ and $\mathcal{IG}$
  - the interface between $\mathcal{OG}$ and $\mathcal{DG}$
  - the interface between $\mathcal{IG}$ and $\mathcal{DG}$

Example: classical square of opposition



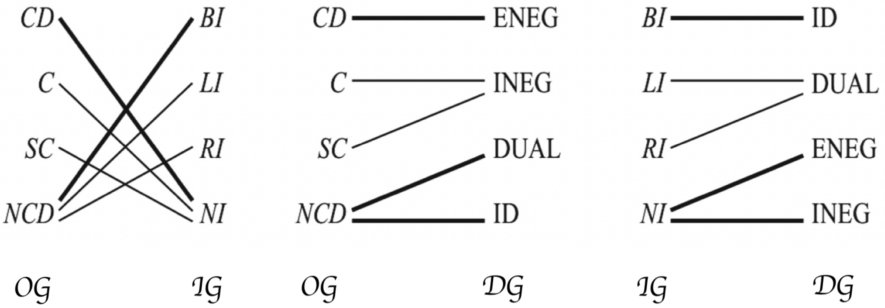


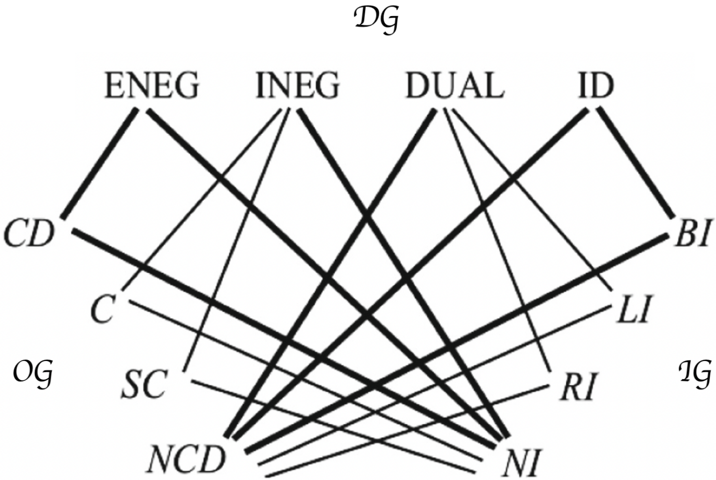




- $\mathcal{OG}/\mathcal{IG}$ multigraph
- $\mathcal{OG}/\mathcal{DG}$ multigraph
- $\mathcal{IG}/\mathcal{DG}$ multigraph

(for the sake of comparison: ADM = $\mathcal{AG}/\mathcal{DG}$ multigraph)





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- we have seen:
    - a $\mathcal{AG}/\mathcal{DG}$ multigraph for the classical square (classical duality)
    - a $\mathcal{AG}/\mathcal{DG}$ multigraph for the Buridan octagon (composed operator duality)
    - $\mathcal{AG}/\mathcal{DG}$ multigraphs for the Moretti and Keynes-Johnson octagons (generalized Post duality)
    - a $\mathcal{OG}/\mathcal{IG}/\mathcal{DG}$ multigraph for the classical square (classical duality)

- what about:
    - a $\mathcal{OG}/\mathcal{IG}/\mathcal{DG}$ multigraph for the Buridan octagon (composed operator duality)
    - $\mathcal{OG}/\mathcal{IG}/\mathcal{DG}$ multigraphs for the Moretti and Keynes-Johnson octagons (generalized Post duality)

Thank you! Questions?

More info: `www.logicalgeometry.org`