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Introduction to Logical Geometry

4. Abstract-Logical Properties of Aristotelian Diagrams, Part II

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ESSLLI 2018, Sofia

1. Basic Concepts and Bitstring Semantics
2. Abstract-Logical Properties of Aristotelian Diagrams, Part I
   ☞ Aristotelian, Opposition, Implication and Duality Relations
3. Visual-Geometric Properties of Aristotelian Diagrams
   ☞ Informational Equivalence, Symmetry and Distance
4. **Abstract-Logical Properties of Aristotelian Diagrams, Part II**
   ☞ **Boolean Structure and Logic-Sensitivity**
5. Case Studies and Philosophical Outlook

- recall from lecture 1:
  - since the Aristotelian relations are defined in purely Boolean terms, the Aristotelian structure of a fragment is entirely determined by its Boolean structure
  - if two fragments have the same Boolean structure, they also have the same Aristotelian structure
  - every Boolean isomorphism between two fragments is also an Aristotelian isomorphism between those fragments

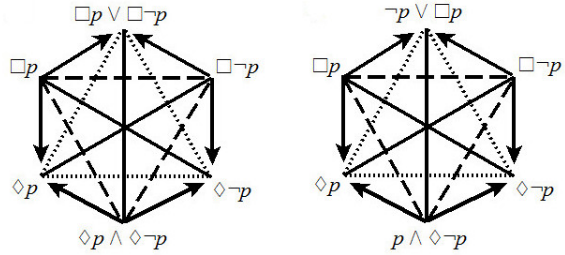
- the inverse does **not** hold:
  - the Boolean structure of a fragment is not entirely determined by its Aristotelian structure
  - it is perfectly possible for two fragments to have the same Aristotelian structure, and yet different Boolean structure
  - there exist Aristotelian isomorphisms between two fragments that are not Boolean isomorphisms between those two fragments

Example

- easiest + oldest example of this phenomenon (Pellissier 2008)
- two hexagons in the modal logic S5
- bijection f between the two hexagons:

1. $f(\Box p) = \Box p$	3. $f(\Box\neg p) = \Box\neg p$	5. $f(\Box p \vee \Box\neg p) = \neg p \vee \Box p$
2. $f(\Diamond p) = \Diamond p$	4. $f(\Diamond\neg p) = \Diamond\neg p$	6. $f(\Diamond p \wedge \Diamond\neg p) = p \wedge \Diamond\neg p$

- f is clearly an Aristotelian isomorphism (check visually!)



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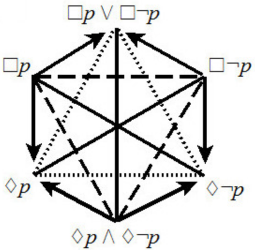
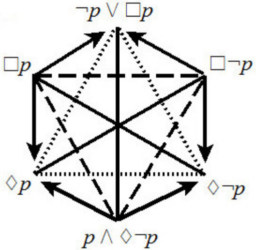
- since the two hexagons are **Aristotelian isomorphic**, they belong to the **same Aristotelian family**
- specifically: both diagrams are JSB hexagons

- nevertheless, they have clear **Boolean differences**:

  (1) $\Box p \vee \Box\neg p$ is equivalent to the disjunction of $\Box p$ and $\Box\neg p$, but $\neg p \vee \Box p$ is not equivalent to the disjunction of $\Box p$ and $\Box\neg p$

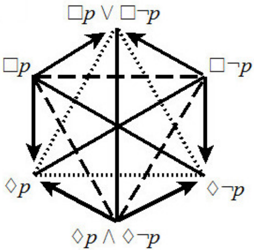


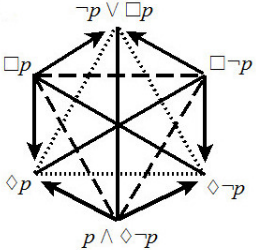


- since the two hexagons are **Aristotelian isomorphic**, they belong to the **same Aristotelian family**
- specifically: both diagrams are JSB hexagons

- nevertheless, they have clear **Boolean differences**:

  (2) $\Diamond p \wedge \Diamond\neg p$ is equivalent to the conjunction of $\Diamond p$ and $\Diamond\neg p$,
  but $p \wedge \Diamond\neg p$ is not equivalent to the conjunction of $\Diamond p$ and $\Diamond\neg p$

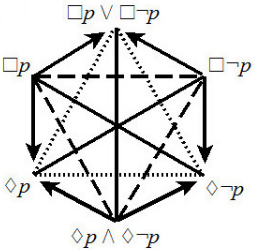


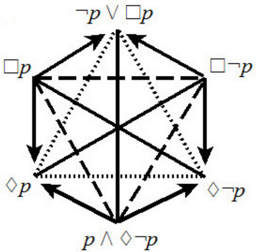


- since the two hexagons are **Aristotelian isomorphic**, they belong to the **same Aristotelian family**
- specifically: both diagrams are JSB hexagons

- nevertheless, they have clear **Boolean differences**:

    (3) the disjunction of $\Box p$, $\Box\neg p$ and $\Diamond p \wedge \Diamond\neg p$ is a tautology, but the disjunction of $\Box p$, $\Box\neg p$ and $p \wedge \Diamond\neg p$ is not a tautology

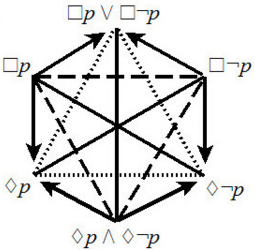


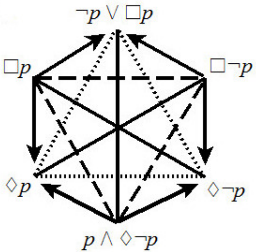


- since the two hexagons are **Aristotelian isomorphic**, they belong to the **same Aristotelian family**
- specifically: both diagrams are JSB hexagons

- nevertheless, they have clear **Boolean differences**:

  (4) the conjunction of $\Diamond p$, $\Diamond\neg p$ and $\Box p \vee \Box\neg p$ is a contradiction, but the conjunction of $\Diamond p$, $\Diamond\neg p$ and $\neg p \vee \Box p$ is not a contradiction





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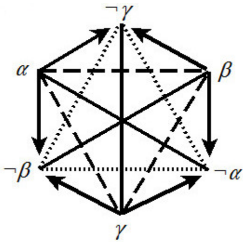
- **generic description** of a JSB hexagon
  (independent of concrete formulas, logical system, etc.)
  - fragment $\mathcal{F}_{JSB} = \{\alpha, \beta, \gamma, \neg\alpha, \neg\beta, \neg\gamma\}$
  - the formulas α, β, γ are pairwise contrary

- the Aristotelian family of JSB hexagons has **two Boolean subtypes**:
  - **strong** JSB hexagon: $\alpha \vee \beta \vee \gamma$ is a tautology
  - **weak** JSB hexagon: $\alpha \vee \beta \vee \gamma$ is not a tautology



- consider different Boolean subtypes of some given Aristotelian family

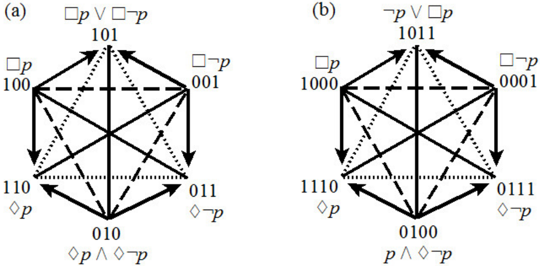
- **different Boolean subtypes** have **different Boolean properties**
- e.g. strong vs. weak JSB hexagon $\Rightarrow$ at least 4 Boolean differences

- these differences can be summarized as follows:
  different Boolean subtypes have **different Boolean closures**,
  or more specifically: Boolean closures of different sizes

- recall that bitstring length measures the size of the Boolean closure

- **different Boolean subtypes** are encoded by means of **bitstrings of different lengths**

- our example of a **strong** JSB hexagon
  - induces the partition $\Pi_{strong} := \{\Box p, \Diamond p \wedge \Diamond\neg p, \Box\neg p\}$ of S5
  - $|\Pi_{strong}| = 3 \Rightarrow$ **bitstrings of length 3**
  - Boolean closure: $2^3 = 8$ elements, of which $2^3 - 2 = 6$ are contingent
- our example of a **weak** JSB hexagon
  - induces the partition $\Pi_{weak} := \{\Box p, p \wedge \Diamond\neg p, \neg p \wedge \Diamond p, \Box\neg p\}$ of S5
  - $|\Pi_{weak}| = 4 \Rightarrow$ **bitstrings of length 4**
  - Boolean closure: $2^4 = 16$ elements, of which $2^4 - 2 = 14$ are contingent



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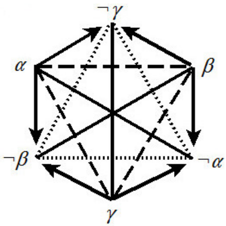
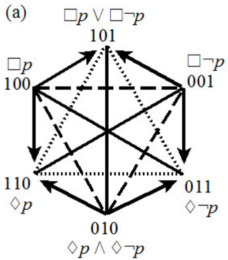
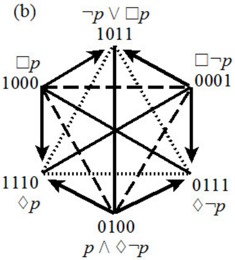
- this bitstring analysis summarizes all the individual Boolean differences

strong JSB	bitstrings of **length 3**	**weak**	bitstrings of **length 4**
$\neg\gamma \equiv \alpha \vee \beta$	$101 = 100 \vee 001$	$\not\equiv$	$1011 \neq 1000 \vee 0001$
$\gamma \equiv \neg\alpha \wedge \neg\beta$	$010 = 011 \wedge 110$	$\not\equiv$	$0100 \neq 0111 \wedge 1110$
$\alpha \vee \beta \vee \gamma \equiv \top$	$100 \vee 001 \vee 010 = 111$	$\not\equiv$	$1000 \vee 0001 \vee 0100 \neq 1111$
$\neg\alpha \wedge \neg\beta \wedge \neg\gamma \equiv \bot$	$011 \wedge 110 \wedge 101 = 000$	$\not\equiv$	$0111 \wedge 1110 \wedge 1011 \neq 0000$





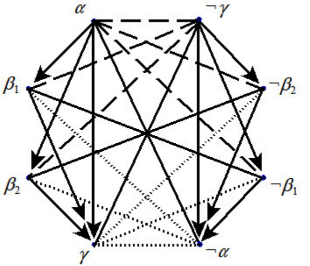


Boolean subtypes of the Buridan octagons

- generic description of a **Buridan octagon**:
  - fragment $\mathcal{F}_{Buri} = \{\alpha, \beta_1, \beta_2, \gamma, \neg\alpha, \neg\beta_1, \neg\beta_2, \neg\gamma\}$
  - subalternations from α to β_1, β_2 to γ; unconnected β_1, β_2

- the Buridan octagons come in **three Boolean subtypes**:

strong Buridan octagon	$\alpha \equiv \beta_1 \wedge \beta_2$ and $\gamma \equiv \beta_1 \vee \beta_2$	**length 4**
intermediate Buridan octagon	exactly one equivalence	**length 5**
weak Buridan hexagon	neither equivalence	**length 6**



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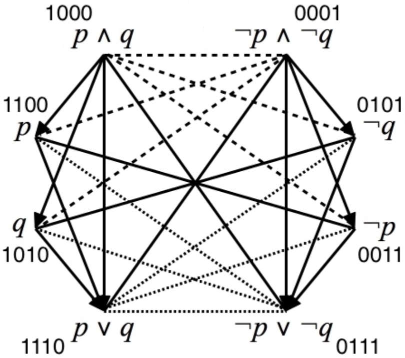
Example of a strong Buridan octagon

- induces the partition $\Pi_{strong} = \{p \wedge q, p \wedge \neg q, \neg p \wedge q, \neg p \wedge \neg q\}$
- $|\Pi_{strong}| = 4 \Rightarrow$ bitstrings of length 4

- $p \wedge q$ is equivalent to the conjunction of p and q $(1000 = 1100 \wedge 1010)$
- $p \vee q$ is equivalent to the disjunction of p and q $(1110 = 1100 \vee 1010)$



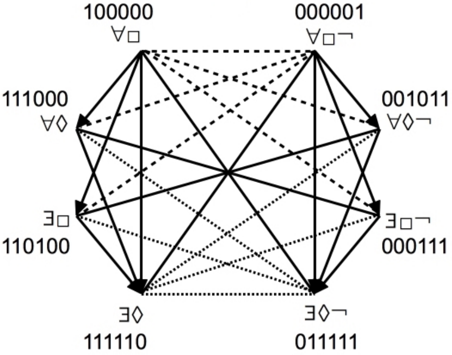
- fragment of 8 *de re* modal formulas (with ampliation):

1. all S are necessarily P	$\exists x \Diamond Sx \wedge \forall x(\Diamond Sx \rightarrow \Box Px)$	$\forall\Box$
2. all S are possibly P	$\exists x \Diamond Sx \wedge \forall x(\Diamond Sx \rightarrow \Diamond Px)$	$\forall\Diamond$
3. some S are necessarily P	$\exists x(\Diamond Sx \wedge \Box Px)$	$\exists\Box$
4. some S are possibly P	$\exists x(\Diamond Sx \wedge \Diamond Px)$	$\exists\Diamond$
5. all S are necessarily not P	$\forall x(\Diamond Sx \rightarrow \Box\neg Px)$	$\forall\Box\neg$
6. all S are possibly not P	$\forall x(\Diamond Sx \rightarrow \Diamond\neg Px)$	$\forall\Diamond\neg$
7. some S are necessarily not P	$\neg\exists x \Diamond Sx \vee \exists x(\Diamond Sx \wedge \Box\neg Px)$	$\exists\Box\neg$
8. some S are possibly not P	$\neg\exists x \Diamond Sx \vee \exists x(\Diamond Sx \wedge \Diamond\neg Px)$	$\exists\Diamond\neg$

- this fragment induces the following partition:

$$\Pi_{weak} = \{ \forall\Box,\ \forall\Diamond \wedge \exists\Box \wedge \exists\Diamond\neg,\ \forall\Diamond \wedge \forall\Diamond\neg,\ \exists\Box \wedge \exists\Box\neg,\ \forall\Diamond\neg \wedge \exists\Box\neg \wedge \exists\Diamond,\ \forall\Box\neg \}$$

- $|\Pi_{weak}| = 6 \Rightarrow$ bitstrings of length 6
- $\forall\Box \not\equiv \forall\Diamond \wedge \exists\Box$ $(100000 \neq 111000 \wedge 110100)$
- $\exists\Diamond \not\equiv \forall\Diamond \vee \exists\Box$ $(111110 \neq 111000 \vee 110100)$



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Example of an intermediate Buridan octagon

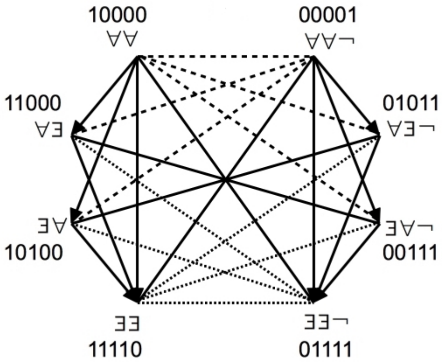
- fragment of 8 propositions 'of unusual construction':

1. all S all P are	$\exists xSx \wedge \exists yPy \wedge \forall x(Sx \rightarrow \forall y(Py \rightarrow x = y))$	$\forall\forall$
2. all S some P are	$\exists xSx \wedge \forall x(Sx \rightarrow \exists y(Py \wedge x = y))$	$\forall\exists$
3. some S all P are	$\exists yPy \wedge \exists x(Sx \wedge \forall y(Py \rightarrow x = y))$	$\exists\forall$
4. some S some P are	$\exists x(Sx \wedge \exists y(Py \wedge x = y))$	$\exists\exists$
5. all S all P are not	$\forall x(Sx \rightarrow \forall y(Py \rightarrow x \neq y))$	$\forall\forall\neg$
6. all S some P are not	$\neg\exists yPy \vee \forall x(Sx \rightarrow \exists y(Py \wedge x \neq y))$	$\forall\exists\neg$
7. some S all P are not	$\neg\exists xSx \vee \exists x(Sx \wedge \forall y(Py \rightarrow x \neq y))$	$\exists\forall\neg$
8. some S some P are not	$\neg\exists xSx \vee \neg\exists yPy \vee \exists x(Sx \wedge \exists y(Py \wedge x \neq y))$	$\exists\exists\neg$

- this fragment induces the following partition:

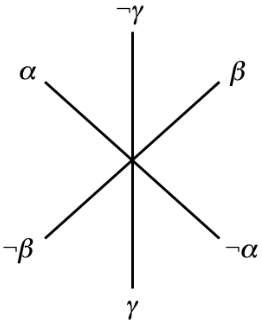
$$\Pi_{intermediate} = \{ \forall\forall, \quad \forall\exists \wedge \forall\exists\neg, \quad \exists\forall \wedge \exists\forall\neg, \quad \forall\exists\neg \wedge \exists\forall\neg \wedge \exists\exists, \quad \forall\forall\neg \}$$

- $|\Pi_{intermediate}| = 5 \Rightarrow$ bitstrings of length 5
- $\forall\forall \equiv \forall\exists \wedge \exists\forall$ $\quad (10000 = 11000 \wedge 10100)$
- $\exists\exists \not\equiv \forall\exists \vee \exists\forall$ $\quad (11110 \neq 11000 \vee 10100)$



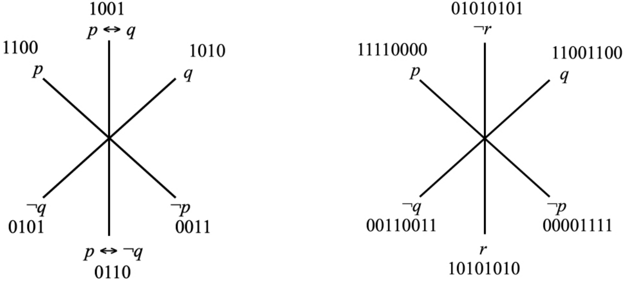
- generic description of a **U12 hexagon**:
  - $\mathcal{F}_{U12} = \{\alpha, \beta, \gamma, \neg\alpha, \neg\beta, \neg\gamma\}$
  - α, β, γ are pairwise unconnected

- the U12 hexagons come in **five Boolean subtypes**:
  U12 hexagons that require bitstrings of length 4, 5, 6, 7, 8



- left: the fragment $\{p, q, \neg p, \neg q, p \leftrightarrow q, p \leftrightarrow \neg q\}$ induces the partition $\Pi_{left} = \{p \wedge q, p \wedge \neg q, \neg p \wedge q, \neg p \wedge \neg q\}$ $|\Pi_{left}| = 4 \Rightarrow$ length 4

- right: the fragment $\{p, q, r, \neg p, \neg q, \neg r\}$ induces the partition $\Pi_{right} = \{p \wedge q \wedge r, p \wedge q \wedge \neg r, p \wedge \neg q \wedge r, p \wedge \neg q \wedge \neg r, \neg p \wedge q \wedge r, \neg p \wedge q \wedge \neg r, \neg p \wedge \neg q \wedge r, \neg p \wedge \neg q \wedge \neg r\}$ $|\Pi_{right}| = 8 \Rightarrow$ length 8



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- some Aristotelian families do not have multiple Boolean subtypes:
  - they are **Boolean homogeneous**
  - all their members can be encoded using bitstrings of the **same length**

- some examples:
  - the family of PCDs: requires only bitstrings of length 2
  - the family of classical squares: requires only bitstrings of length 3
  - the family of degenerate squares: requires only bitstrings of length 4
  - the family of SC hexagons: requires only bitstrings of length 4
  - the family of Lenzen octagons: requires only bitstrings of length 5

- note:
  - the most well-known family (classical squares) is Boolean homogeneous
  - this might explain why the issue of Boolean subtypes is not very familiar

- what determines whether a given Aristotelian family $\mathcal{A}$ has multiple Boolean subtypes or is rather Boolean homogeneous?

- recall the definition of the partition induced by fragment $\mathcal{F}$ in logic S:

  $\Pi_{\mathsf{S}}(\mathcal{F}) := \{\alpha \in \mathcal{L} \mid \alpha = \pm\varphi_1 \wedge \cdots \wedge \pm\varphi_m$, and α is S-consistent$\}$

  (elements $\alpha \in \Pi_{\mathsf{S}}(\mathcal{F})$ are called **anchor formulas**) ☞ lecture 1

- how to calculate the partition induced by a **generic description** of some Aristotelian family $\mathcal{A}$ (regardless of any concrete logical system):

  $\Pi_{\mathcal{A}} := \{\alpha \in \mathcal{L} \mid \alpha = \pm\varphi_1 \wedge \cdots \wedge \pm\varphi_m$, and α is $\mathcal{A}$-consistent$\}$

- an anchor formula α is **$\mathcal{A}$-consistent** iff it does not contain two conjuncts that are contradictory or contrary according to the generic description of $\mathcal{A}$

- an anchor formula is
  - $\mathcal{A}$-**consistent** iff it does not contain two conjuncts that are contradictory or contrary according to the generic description of $\mathcal{A}$
  - $\mathcal{A}$-**inconsistent** iff it does contain two conjuncts that are contradictory or contrary according to the generic description of $\mathcal{A}$

- **lemma**: if an anchor formula is $\mathcal{A}$-inconsistent, then it is S-inconsistent (contrapositive: if it is S-consistent, then it is $\mathcal{A}$-consistent)

- the converse does **not** hold:
  an anchor formula can be S-inconsistent and yet $\mathcal{A}$-consistent

- concrete example: $(p \vee q) \wedge \neg p \wedge \neg q$
  - this formula is CPL-inconsistent
  - this formula is $\mathcal{A}$-consistent (for any Aristotelian family $\mathcal{A}$)

- we have just seen:
  - if an anchor formula is $\mathcal{A}$-inconsistent, then it is S-inconsistent
  - an anchor formula can be S-inconsistent and yet $\mathcal{A}$-consistent
  - example: $(p \vee q) \wedge \neg p \wedge \neg q$ (**three** conjuncts)

- **lemma**: consider an anchor formula with **at most two** conjuncts:
  - if that anchor formula is $\mathcal{A}$-inconsistent, then it is S-inconsistent
  - if that anchor formula is S-inconsistent, then it is $\mathcal{A}$-inconsistent

  $\Rightarrow$ $\mathcal{A}$-consistency guarantees S-consistency

- **lemma**: consider an anchor formula with **at least three** conjuncts:
  - if that anchor formula is $\mathcal{A}$-inconsistent, then it is S-inconsistent
  - that anchor formula can be S-inconsistent and yet $\mathcal{A}$-consistent

  $\Rightarrow$ $\mathcal{A}$-consistency does not guarantee S-consistency

- what determines whether a given Aristotelian family $\mathcal{A}$ has multiple Boolean subtypes or is rather Boolean homogeneous?

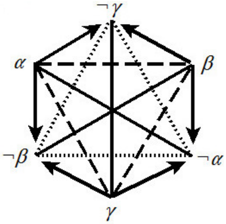
- $\Pi_{\mathcal{A}} = \{\alpha \in \mathcal{L} \mid \alpha = \pm\varphi_1 \wedge \cdots \wedge \pm\varphi_m$, and α is $\mathcal{A}$-consistent$\}$

- each anchor formula $\alpha \in \Pi_{\mathcal{A}}$ is $\mathcal{A}$-consistent
  - if α has at most two conjuncts, it is also guaranteed to be S-consistent
  - if α has at least three conjuncts, it is not guaranteed to be S-consistent

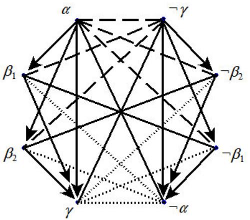
- **case distinction**:
  - all $\alpha \in \Pi_{\mathcal{A}}$ are guaranteed to be S-consistent
    $\Rightarrow \mathcal{A}$ is **Boolean homogeneous**, with single bitstring length $|\Pi_{\mathcal{A}}|$
  - $n > 0$ formulas in $\Pi_{\mathcal{A}}$ are not guaranteed to be S-consistent
    $\Rightarrow \mathcal{A}$ has $n+1$ **Boolean subtypes**, with bitstring lengths
    $|\Pi_{\mathcal{A}}| - n, \ldots, |\Pi_{\mathcal{A}}| - 1, |\Pi_{\mathcal{A}}|$

- anchor formulas in Π_{JSB}:
  - α guaranteed to be S-consistent
  - β guaranteed to be S-consistent
  - γ guaranteed to be S-consistent
  - $\neg\alpha \wedge \neg\beta \wedge \neg\gamma$ **not** guaranteed to be S-consistent

- the Aristotelian family of JSB hexagons has 2 Boolean subtypes:
  - length 3, corresponding to partition $\{\alpha, \beta, \gamma\}$
  - length 4, corresponding to partition $\{\alpha, \beta, \gamma, \neg\alpha \wedge \neg\beta \wedge \neg\gamma\}$

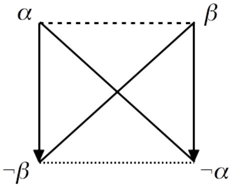


- anchor formulas in Π_{Buri}:
  - α — guaranteed to be S-consistent
  - $\neg\alpha \wedge \beta_1 \wedge \beta_2$ — **not** guaranteed to be S-consistent
  - $\beta_1 \wedge \neg\beta_2$ — guaranteed to be S-consistent
  - $\neg\beta_1 \wedge \beta_2$ — guaranteed to be S-consistent
  - $\neg\beta_1 \wedge \neg\beta_2 \wedge \gamma$ — **not** guaranteed to be S-consistent
  - $\neg\gamma$ — guaranteed to be S-consistent
- the Aristotelian family of Buridan octagons has 3 Boolean subtypes: length 6, length 5, length 4

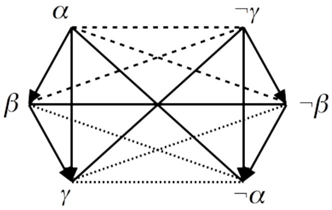


Example 3: classical square

- anchor formulas in Π_{class_sq}:
  - α guaranteed to be S-consistent
  - β guaranteed to be S-consistent
  - $\neg\alpha \wedge \neg\beta$ guaranteed to be S-consistent
- the Aristotelian family of classical squares is Boolean homogeneous (length 3)



- anchor formulas in Π_{SC}:
  - α — guaranteed to be S-consistent
  - $\neg\alpha \wedge \beta$ — guaranteed to be S-consistent
  - $\neg\beta \wedge \gamma$ — guaranteed to be S-consistent
  - $\neg\gamma$ — guaranteed to be S-consistent
- the Aristotelian family of SC hexagons is Boolean homogeneous (length 4)



- anchor formulas in Π_{U12}:
  - $\alpha \wedge \beta \wedge \gamma$ not guaranteed to be S-consistent
  - $\alpha \wedge \beta \wedge \neg\gamma$ not guaranteed to be S-consistent
  - $\alpha \wedge \neg\beta \wedge \gamma$ not guaranteed to be S-consistent
  - $\alpha \wedge \neg\beta \wedge \neg\gamma$ not guaranteed to be S-consistent
  - $\neg\alpha \wedge \beta \wedge \gamma$ not guaranteed to be S-consistent
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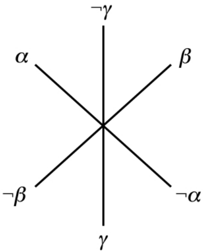

Exceptional case: U12 hexagon

- anchor formulas in Π_{U12}:
  - $\alpha \wedge \beta \wedge \gamma$ — not guaranteed to be S-consistent
  - $\alpha \wedge \beta \wedge \neg\gamma$ — not guaranteed to be S-consistent
  - $\alpha \wedge \neg\beta \wedge \gamma$ — not guaranteed to be S-consistent
  - $\alpha \wedge \neg\beta \wedge \neg\gamma$ — not guaranteed to be S-consistent
  - $\neg\alpha \wedge \beta \wedge \gamma$ — not guaranteed to be S-consistent
  - $\neg\alpha \wedge \beta \wedge \neg\gamma$ — not guaranteed to be S-consistent
  - $\neg\alpha \wedge \neg\beta \wedge \gamma$ — not guaranteed to be S-consistent
  - $\neg\alpha \wedge \neg\beta \wedge \neg\gamma$ — not guaranteed to be S-consistent
- misguided prediction: the Aristotelian family of U12 hexagons has 9 Boolean subtypes: length 8, 7, 6, 5, 4, 3, 2, 1, 0
- **but** encoding unconnectedness requires bitstrings of length at least 4

  ☞ lecture 2
- correct analysis: the Aristotelian family of U12 hexagons has 5 Boolean subtypes: length 8, 7, 6, 5, 4

Broader context

- ongoing research effort in logical geometry:
  develop a **systematic typology** of Aristotelian diagrams
- for each diagram size:
  what are the **Aristotelian families** with that size?
- for each Aristotelian family:
  what are the **Boolean subfamilies** of that Aristotelian family?

		diagram size											
		PCD (1)	square (2)		hexagon (5)					octagon (18)			
bitstring length	2	PCD											
	3		class.		JSB								
	4			degen.	JSB	SC	U4		U12	Buridan		Moretti	...
	5						U4	U8	U12	Buridan	Lenzen	Moretti	...
	6							U8	U12	Buridan			...
	7								U12				...
	8								U12				...
	⋮												

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- recall from lecture 1 (final slide):
  if $\mathcal{F}$ only contains S-contingent formulas and is closed under negation,
  then $\lceil \log_2(|\mathcal{F}|+2) \rceil \quad \leq \quad |\Pi_S(\mathcal{F})| \quad \leq \quad 2^{|\mathcal{F}|/2}$
- some specific cases:
  - $|\mathcal{F}| = 2 \quad \Rightarrow \quad 2 = \lceil \log_2(2+2) \rceil \quad \leq \quad |\Pi_S(\mathcal{F})| \quad \leq \quad 2^{2/2} = 2$
  - $|\mathcal{F}| = 4 \quad \Rightarrow \quad 3 = \lceil \log_2(4+2) \rceil \quad \leq \quad |\Pi_S(\mathcal{F})| \quad \leq \quad 2^{4/2} = 4$
  - $|\mathcal{F}| = 6 \quad \Rightarrow \quad 3 = \lceil \log_2(6+2) \rceil \quad \leq \quad |\Pi_S(\mathcal{F})| \quad \leq \quad 2^{6/2} = 8$
  - $|\mathcal{F}| = 8 \quad \Rightarrow \quad 4 = \lceil \log_2(8+2) \rceil \quad \leq \quad |\Pi_S(\mathcal{F})| \quad \leq \quad 2^{8/2} = 16$

		diagram size											
		PCD (1)	square (2)		hexagon (5)					octagon (18)			
bitstring length	2	PCD											
	3		class.		JSB								
	4			degen.	JSB	SC	U4		U12	Buridan		Moretti	...
	5						U4	U8	U12	Buridan	Lenzen	Moretti	...
	6							U8	U12	Buridan			...
	7								U12				...
	8								U12				...
	⋮												

- recall from lecture 1 (final slide):
  if $\mathcal{F}$ only contains S-contingent formulas and is closed under negation,
  then $2\lceil \log_2(|\Pi_S(\mathcal{F})|) \rceil \quad \leq \quad |\mathcal{F}| \quad \leq \quad 2^{|\Pi_S(\mathcal{F})|} - 2$
- some specific cases:
  - $|\Pi_S(\mathcal{F})| = 2 \quad \Rightarrow \quad 2 = 2\lceil \log_2(2) \rceil \quad \leq \quad |\mathcal{F}| \quad \leq \quad 2^2 - 2 = 2$
  - $|\Pi_S(\mathcal{F})| = 3 \quad \Rightarrow \quad 4 = 2\lceil \log_2(3) \rceil \quad \leq \quad |\mathcal{F}| \quad \leq \quad 2^3 - 2 = 6$
  - $|\Pi_S(\mathcal{F})| = 4 \quad \Rightarrow \quad 4 = 2\lceil \log_2(4) \rceil \quad \leq \quad |\mathcal{F}| \quad \leq \quad 2^4 - 2 = 14$
  - $|\Pi_S(\mathcal{F})| = 5 \quad \Rightarrow \quad 6 = 2\lceil \log_2(5) \rceil \quad \leq \quad |\mathcal{F}| \quad \leq \quad 2^5 - 2 = 30$

	diagram size											
bitstring length	PCD (1)	square (2)		hexagon (5)					octagon (18)			
2	PCD											
3		class.		JSB								
4			degen.	JSB	SC	U4		U12	Buridan		Moretti	...
5						U4	U8	U12	Buridan	Lenzen	Moretti	...
6							U8	U12	Buridan			...
7								U12				...
8								U12				...
⋮												

1. Basic Concepts and Bitstring Semantics

2. Abstract-Logical Properties of Aristotelian Diagrams, Part I
   ☞ Aristotelian, Opposition, Implication and Duality Relations

3. Visual-Geometric Properties of Aristotelian Diagrams
   ☞ Informational Equivalence, Symmetry and Distance

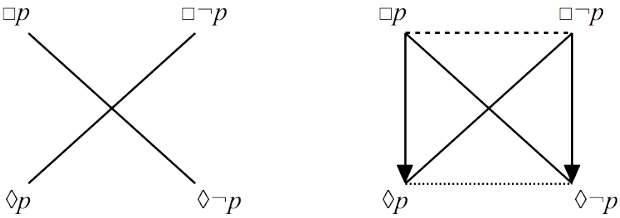
4. **Abstract-Logical Properties of Aristotelian Diagrams, Part II**
   ☞ **Boolean Structure and Logic-Sensitivity**

5. Case Studies and Philosophical Outlook

- Aristotelian diagrams are (in various ways) **sensitive** to the specific details of the underlying logical system
  - informal definition: 'φ and ψ cannot be true together'
  - model-theoretic definition: $\models_S \neg(\varphi \wedge \psi)$
- this point has recently also been emphasized by Claudio Pizzi:
  - "An obvious but frequently neglected proviso concerning the squares of oppositions is that the relations which are claimed to hold between the formulas of the square only subsist with reference to some given background system" (2016)
  - "an ordered 4-[tu]ple is an Aristotelian square always with respect to some system S" (2017)

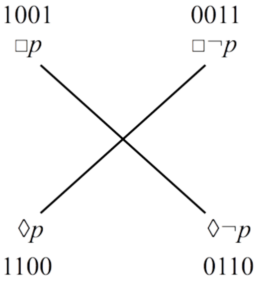
- one fragment $\mathcal{F} := \{\Box p, \Diamond p, \Box\neg p, \Diamond\neg p\}$
- two logical systems:
  - K: basic normal modal logic (all Kripke models)
  - KD: axiom $\Box p \to \Diamond p$ (or $\Diamond\top$) (serial Kripke models)

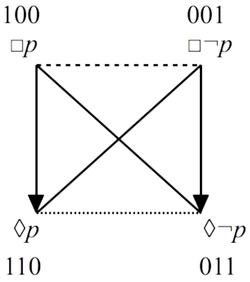
- in K, the fragment $\mathcal{F}$ gives rise to a **degenerate square**
- in KD, the fragment $\mathcal{F}$ gives rise to a **classical square**



- $\Pi_{\mathsf{K}}(\mathcal{F}) = \{\Box p \wedge \Diamond p, \Diamond p \wedge \Diamond\neg p, \Box\neg p \wedge \Diamond\neg p, \Box p \wedge \Box\neg p\}$ (length 4)
- $\Pi_{\mathsf{KD}}(\mathcal{F}) = \{\Box p, \Diamond p \wedge \Diamond\neg p, \Box\neg p\}$ (length 3)

- from K to KD: **delete the fourth bit position**
  - $\Box p \wedge \Box\neg p$ is K-consistent, but KD-inconsistent
  - $\Pi_{\mathsf{KD}}(\mathcal{F}) = \{\alpha \in \Pi_{\mathsf{K}}(\mathcal{F}) \mid \alpha$ is KD-consistent$\}$





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- Aristotelian relations are logic-sensitive:
  - $\Box p$ and $\Box\neg p$ are K-unconnected, but KD-contrary
  - $\Box p$ and $\Diamond p$ are K-unconnected, but in KD-subalternation

- duality relations are not (or rather: less) logic-sensitive:
  - $\Box p$ and $\Box\neg p$ are each other's internal negation, in K as well as KD
  - $\Box p$ and $\Diamond p$ are each other's dual, in K as well as KD

- duality relations **are** sensitive to the underlying logic, but only to **non-Boolean** aspects
  - K and KD are both classical Boolean logics $\Rightarrow$ no differences in duality
  - e.g. $p \wedge q$ and $p \vee q$ are dual in classical logic, but not in intuitionistic logic

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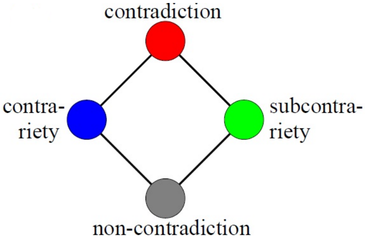
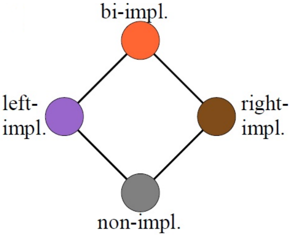
- recall from lecture 2:
  - $\mathcal{AG}_{\mathsf{S}}$ is hybrid between $\mathcal{OG}_{\mathsf{S}}$ and $\mathcal{IG}_{\mathsf{S}}$
  - non-contradiction: $NCD_{\mathsf{S}}(\varphi, \psi)$ iff $\not\models_{\mathsf{S}} \neg(\varphi \wedge \psi)$ and $\not\models_{\mathsf{S}} \neg(\neg\varphi \wedge \neg\psi)$
- consider two logics W,S, and assume that for all φ: $\models_{\mathsf{W}} \varphi \;\Rightarrow\; \models_{\mathsf{S}} \varphi$
- **theorem**: from the weaker logic to the stronger logic
  - if $CD_{\mathsf{W}}(\varphi, \psi)$ then $CD_{\mathsf{S}}(\varphi, \psi)$
  - if $C_{\mathsf{W}}(\varphi, \psi)$ then $C_{\mathsf{S}}(\varphi, \psi)$ or $CD_{\mathsf{S}}(\varphi, \psi)$
  - if $SC_{\mathsf{W}}(\varphi, \psi)$ then $SC_{\mathsf{S}}(\varphi, \psi)$ or $CD_{\mathsf{S}}(\varphi, \psi)$
  - if $NCD_{\mathsf{W}}(\varphi, \psi)$ then $NCD_{\mathsf{S}}(\varphi, \psi)$ or $C_{\mathsf{S}}(\varphi, \psi)$ or $SC_{\mathsf{S}}(\varphi, \psi)$ or $CD_{\mathsf{S}}(\varphi, \psi)$
- **theorem**: from the stronger logic to the weaker logic
  - if $CD_{\mathsf{S}}(\varphi, \psi)$ then $CD_{\mathsf{W}}(\varphi, \psi)$ or $C_{\mathsf{W}}(\varphi, \psi)$ or $SC_{\mathsf{W}}(\varphi, \psi)$ or $NCD_{\mathsf{W}}(\varphi, \psi)$
  - if $C_{\mathsf{S}}(\varphi, \psi)$ then $C_{\mathsf{W}}(\varphi, \psi)$ or $NCD_{\mathsf{W}}(\varphi, \psi)$
  - if $SC_{\mathsf{S}}(\varphi, \psi)$ then $SC_{\mathsf{W}}(\varphi, \psi)$ or $NCD_{\mathsf{W}}(\varphi, \psi)$
  - if $NCD_{\mathsf{S}}(\varphi, \psi)$ then $NCD_{\mathsf{W}}(\varphi, \psi)$

- diagrammatic summary of the two theorems:
  - going to a stronger logic can make you go up in the diagram
  - going to a weaker logic can make you go down in the diagram



- recall the **informativity ordering** $\leq_i^\forall$ of $\mathcal{OG}$ ☞ lecture 2
- **theorem**: the following are equivalent:
  - S is at least as strong as W
  - for all φ, ψ: if $R_\mathsf{W}(\varphi, \psi)$ and $R'_\mathsf{S}(\varphi, \psi)$, then $R \leq_i^\forall R'$

- a completely analogous story can be told about the implication relations
- summary:
  - going to a stronger logic can make you go up in the diagram
  - going to a weaker logic can make you go down in the diagram



- sources of logic-sensitivity in Aristotelian diagrams:
  - logic-sensitivity of the Aristotelian (opp./imp.) **relations** themselves
  - the condition that Aristotelian diagrams only contain **pairwise non-equivalent** formulas
  - the condition that Aristotelian diagrams only contain **contingent** formulas

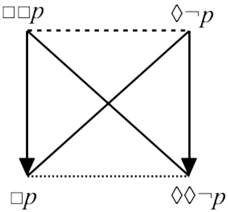
- **equivalence** is a logic-sensitive notion:
  two formulas might be equivalent in one logic,
  and not equivalent in another logic

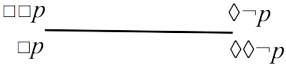
- **contingency** is a logic-sensitive notion:
  a formula might be contingent in one logic, and not in another logic

- one fragment $\mathcal{F} := \{\Box\Box p, \Box p, \Diamond\Diamond\neg p, \Diamond\neg p\}$
- two logical systems:
  - KT: axiom $\Box p \rightarrow p$ (reflexive Kripke models)
  - KT4: axioms $\Box p \rightarrow p$, $\Box p \rightarrow \Box\Box p$ (refl., transitive Kripke models)

- in KT, the fragment $\mathcal{F}$ gives rise to a **classical square**
- in KT4, the fragment $\mathcal{F}$ gives rise to a **PCD**

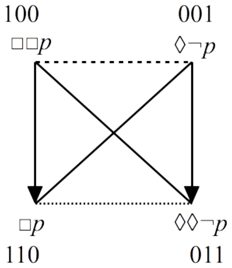
- we go from $C_{\mathsf{KT}}(\Box\Box p, \Diamond\neg p)$ to $CD_{\mathsf{KT4}}(\Box\Box p, \Diamond\neg p)$
- we go from $LI_{\mathsf{KT}}(\Box\Box p, \Box p)$ to $BI_{\mathsf{KT4}}(\Box\Box p, \Box p)$

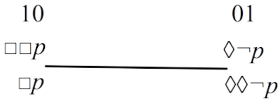




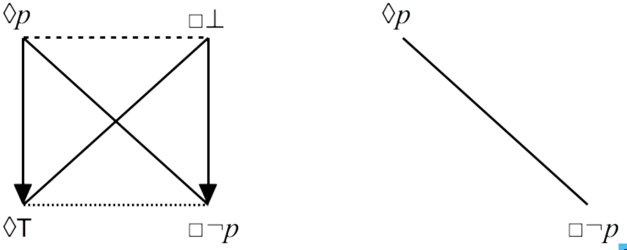
The non-equivalence condition: bitstring analysis

- $\Pi_{\mathsf{KT}}(\mathcal{F}) = \{\Box\Box p, \Box p \wedge \Diamond\Diamond\neg p, \Diamond\neg p\}$ (length 3)
- $\Pi_{\mathsf{KT4}}(\mathcal{F}) = \{\Box p, \Diamond\neg p\}$ (length 2)
- from KT to KT4: **delete the second bit position**
  - $\Box p \wedge \Diamond\Diamond\neg p$ is KT-consistent, but KT4-inconsistent
  - $\Pi_{\mathsf{KT4}}(\mathcal{F}) = \{\alpha \in \Pi_{\mathsf{KT}}(\mathcal{F}) \mid \alpha$ is KT4-consistent$\}$

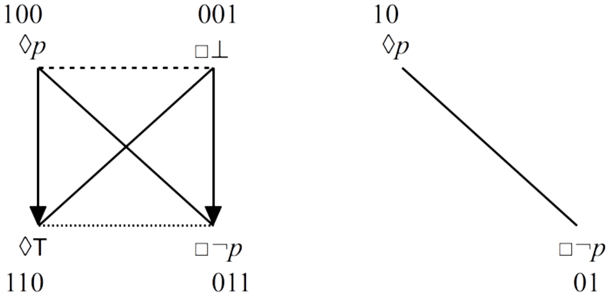




- one fragment $\mathcal{F} := \{\Diamond p, \Diamond\top, \Box\neg p, \Box\bot\}$
- two logical systems:
  - K: basic normal modal logic (all Kripke models)
  - KD: axiom $\Box p \to \Diamond p$ (or $\Diamond\top$) (serial Kripke models)
- in K, the fragment $\mathcal{F}$ gives rise to a **classical square**
- in KD, the fragment $\mathcal{F}$ gives rise to a **PCD**
- $\Diamond\top$ is contingent in K, but a tautology in KD
- $\Box\bot$ is contingent in K, but a contradiction in KD



- $\Pi_{\mathsf{K}}(\mathcal{F}) = \{\Diamond p, \Diamond\top \wedge \Box\neg p, \Box\bot\}$ (length 3)
- $\Pi_{\mathsf{KD}}(\mathcal{F}) = \{\Diamond p, \Box\neg p\}$ (length 2)
- from K to KD: **delete the third bit position**
  - $\Box\bot$ is K-consistent, but KD-inconsistent
  - $\Pi_{\mathsf{KD}}(\mathcal{F}) = \{\alpha \in \Pi_{\mathsf{K}}(\mathcal{F}) \mid \alpha$ is KD-consistent$\}$



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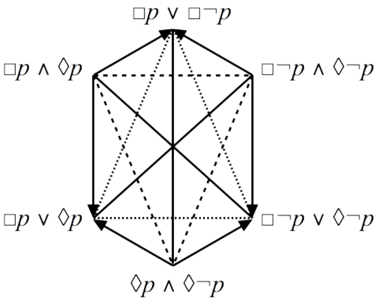
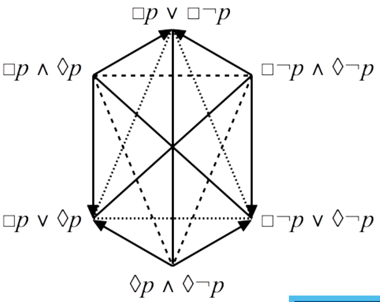
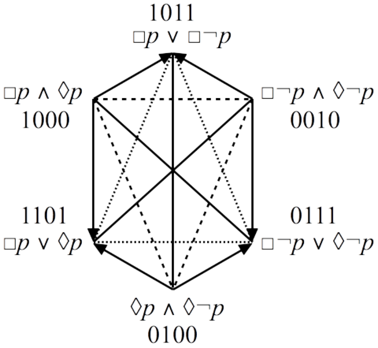
- logic-sensitivity: one fragment, two logics $\Rightarrow$ two different diagrams
  - classical square vs. degenerate square
  - classical square vs. PCD
  - JSB hexagon vs. classical square
  - Buridan octagon vs. Lenzen octagon
- until now: the two diagrams belong to **different Aristotelian families**
- also possible:
  - the two diagrams belong to the **same Aristotelian family**
  - but to **different Boolean subtypes** of that Aristotelian family

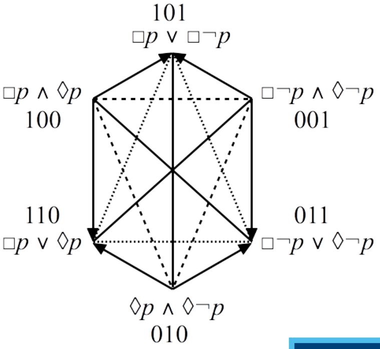
- one fragment $\mathcal{F}$, two logical systems: K and KD
- $\mathcal{F} := \{\Box p \wedge \Diamond p, \Box p \vee \Diamond p, \Box\neg p \wedge \Diamond\neg p, \Box\neg p \vee \Diamond\neg p, \Box p \vee \Box\neg p, \Diamond p \wedge \Diamond\neg p\}$
- in K, the fragment $\mathcal{F}$ gives rise to a **weak JSB hexagon**
- in KD, the fragment $\mathcal{F}$ gives rise to a **strong JSB hexagon**
- $\not\models_{\mathsf{K}} (\Box p \wedge \Diamond p) \vee (\Box\neg p \wedge \Diamond\neg p) \vee (\Diamond p \wedge \Diamond\neg p)$
- $\models_{\mathsf{KD}} (\Box p \wedge \Diamond p) \vee (\Box\neg p \wedge \Diamond\neg p) \vee (\Diamond p \wedge \Diamond\neg p)$



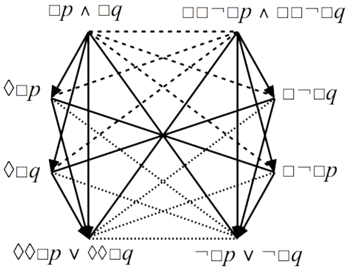


- $\Pi_{\mathsf{K}}(\mathcal{F}) = \{\Box p \wedge \Diamond p, \Diamond p \wedge \Diamond\neg p, \Box\neg p \wedge \Diamond\neg p, \varphi_{long}\}$ (length 4)
- $\Pi_{\mathsf{KD}}(\mathcal{F}) = \{\Box p, \Diamond p \wedge \Diamond\neg p, \Box\neg p\}$ (length 3)
- $\varphi_{long} := (\Box p \vee \Diamond p) \wedge (\Box\neg p \vee \Diamond\neg p) \wedge (\Box p \vee \Box\neg p)$
- from K to KD: **delete the fourth bit position**
  (φ_{long} is K-consistent, but KD-inconsistent)





- one fragment $\mathcal{F}$, three logics: KT, KT4 (= S4) and KT45 (= S5)
- $\mathcal{F} := \{\Box p \wedge \Box q, \Diamond\Box p, \Diamond\Box q, \Diamond\Diamond\Box p \vee \Diamond\Diamond\Box q\}$ (+ negations)
- in KT, the fragment $\mathcal{F}$ gives rise to a **weak Buridan octagon**
- in KT4, the fragment $\mathcal{F}$ gives rise to an **intermediate Buridan octagon**
- in KT45, the fragment $\mathcal{F}$ gives rise to a **strong Buridan octagon**
- $\Box p \wedge \Box q \not\equiv_{\mathsf{KT}} \Diamond\Box p \wedge \Diamond\Box q$ and $\Diamond\Diamond\Box p \vee \Diamond\Diamond\Box q \not\equiv_{\mathsf{KT}} \Diamond\Box p \vee \Diamond\Box q$
- $\Box p \wedge \Box q \not\equiv_{\mathsf{KT4}} \Diamond\Box p \wedge \Diamond\Box q$ and $\Diamond\Diamond\Box p \vee \Diamond\Diamond\Box q \equiv_{\mathsf{KT4}} \Diamond\Box p \vee \Diamond\Box q$
- $\Box p \wedge \Box q \equiv_{\mathsf{KT45}} \Diamond\Box p \wedge \Diamond\Box q$ and $\Diamond\Diamond\Box p \vee \Diamond\Diamond\Box q \equiv_{\mathsf{KT45}} \Diamond\Box p \vee \Diamond\Box q$



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- many different types of logic-sensitivity:
  - based on the Aristotelian relations
  - based on the diagrammatic condition of non-equivalence
  - based on the diagrammatic condition of contingency
  - based on Boolean subtypes

- there are many **cross-connections** among these different types

- example:
  - for any 4-formula fragment $\mathcal{F} = \{\varphi, \psi, \neg\varphi, \neg\psi\}$, define a 6-formula fragment $H(\mathcal{F}) := \{\varphi \wedge \psi, \varphi \vee \psi, \neg\varphi \wedge \neg\psi, \neg\varphi \vee \neg\psi, \varphi \vee \neg\psi, \neg\varphi \wedge \psi\}$
  - **theorem**: for any logical system S:
    - if $\mathcal{F}$ is a degenerate square in S, then $H(\mathcal{F})$ is a weak JSB hexagon in S
    - if $\mathcal{F}$ is a classical square in S, then $H(\mathcal{F})$ is a strong JSB hexagon in S

Thank you!

Questions?

More info: `www.logicalgeometry.org`