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Introduction to Logical Geometry

3. Visual-Geometric Properties of Aristotelian Diagrams

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1. Basic Concepts and Bitstring Semantics

2. Abstract-Logical Properties of Aristotelian Diagrams, Part I
   ☞ Aristotelian, Opposition, Implication and Duality Relations

3. **Visual-Geometric Properties of Aristotelian Diagrams**
   ☞ **Informational Equivalence, Symmetry and Distance**

4. Abstract-Logical Properties of Aristotelian Diagrams, Part II
   ☞ Boolean Structure and Logic-Sensitivity

5. Case Studies and Philosophical Outlook

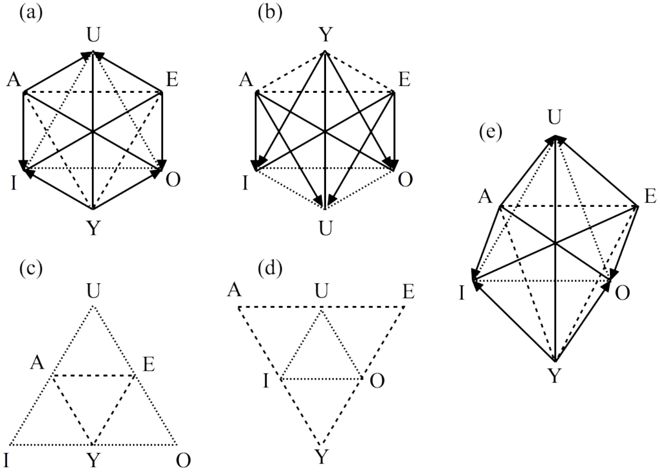
- Aristotelian diagrams represent logical structure/information
  - Aristotelian relations
    - classical square: 2 *CD*, 1 *C*, 1 *SC*, 2 *SA*
    - degenerate square: 2 *CD*
  - underlying Boolean structure
    - classical square: Boolean closure is (isomorphic to) $\mathbb{B}_3$
    - degenerate square: Boolean closure is (isomorphic to) $\mathbb{B}_4$

- diagrams belonging to different Aristotelian families are **not informationally equivalent**
  - they visualize different logical structures
  - differences between diagrams $\leftrightsquigarrow$ differences between logical structures

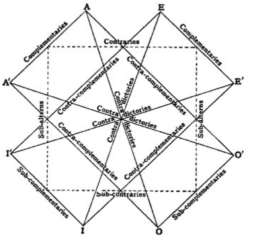
- Jill Larkin and Herbert Simon (1987), *Why a Diagram is (Sometimes) Worth 10.000 Words*

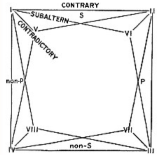
- if we focus on diagrams belonging to the **same Aristotelian family**, we notice that different authors still use **vastly different diagrams**:
  - logical properties of the diagram are fully determined
  - visual-geometric properties are still seriously underspecified
    $\Rightarrow$ various design choices possible

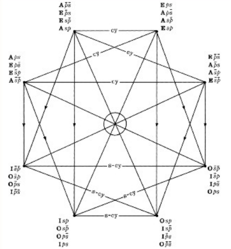
- multiple diagrams for the same formulas and logical system are:
  - **informationally equivalent**
    - contain the same logical information
    - visualize one and the same logical structure
  - **not** necessarily **computationally/cognitively equivalent**:
    - one diagram might be more helpful/useful than the other ones (ease of access to the information contained in the diagram)
    - visual differences might influence diagrams' effectiveness (user comprehension of the underlying logical structure)

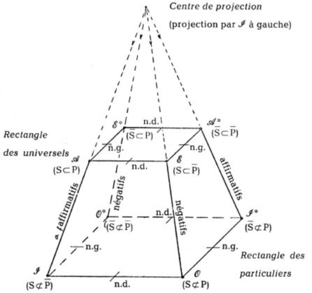


standard and alternative visualisations of the JSB family









standard and alternative visualisations of the Keynes-Johnson family

How to choose among informationally equivalent diagrams?

$\Rightarrow$ rely on general cognitive principles (Corin Gurr, Barbara Tversky):

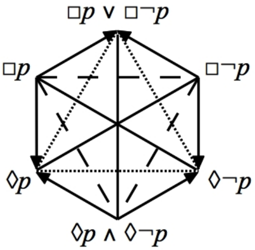
- information selection/ommission and simplification/distortion
- **Apprehension Principle**: the content/structure of the visualization can readily and correctly be perceived and understood
- **Congruence Principle**: the content/structure of the visualization corresponds to the content/structure of the desired mental representation

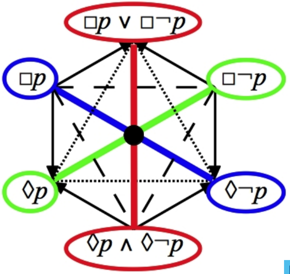
[abstract-logical]		[visual-geometric]
properties, relations among sets of formulas	$\longleftarrow$ **isomorphism congruence** $\longrightarrow$	shape characteristics of the diagrams

- a good diagram simultaneously engages the user's **logical** and **visual** cognitive systems
- facilitate inferential or heuristic **free rides** (Atsushi Shimojima)
  - logical properties are directly manifested in the diagram's visual features
  - user can grasp these properties with little cognitive effort
    $\Rightarrow$ "*you don't have to reason about it, you just see it!*"
- suppose that Aristotelian diagrams D1 and D2 have differerent shapes:
  - shape of D1 more clearly isomorphic to subject matter
  - shape of D2 less clearly isomorphic to subject matter
- then D1 will trigger more heuristics than D2:
  - ceteris paribus, D1 will be a more effective visualization than D2
  - D1 and D2 are not computationally/cognitively equivalent

- two assumptions (satisfied by nearly all diagrams in the literature):
  - the fragment is **closed under negation** (if $\varphi \in \mathcal{F}$ then $\neg\varphi \in \mathcal{F}$)
  - negation is visualized by means of **central symmetry** (φ and $\neg\varphi$ occupy diametrically opposed points in the diagram)

- since the fragment is closed under negation, it can be seen
  - as consisting of $2n$ formulas
  - as consisting of n pairs of contradictory formulas (PCDs)



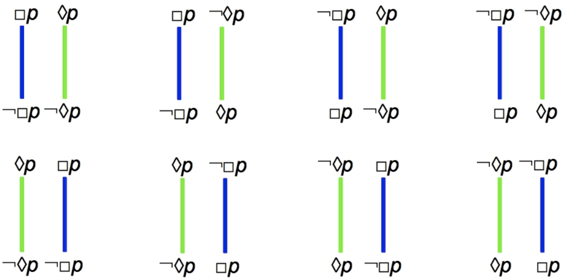


Configurations of PCDs

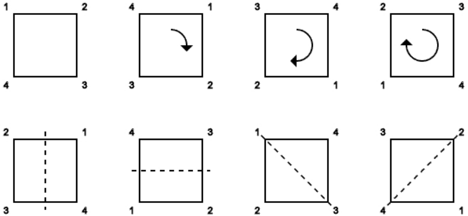
- number of configurations of n PCDs: $2^n \times n!$
  - the n PCDs can be ordered in $n!$ different ways
  - each of the n PCDs has 2 orientations: $(\varphi, \neg\varphi)$ vs. $(\neg\varphi, \varphi)$

- strictly based on the **logical** properties of the fragment
- independent of any concrete visualization

- example: for $n = 2$ PCDs, there are $2^n \times n! = 8$ configurations



- polygon/polyhedron $\mathcal{P}$ to visualize an n-PCD logical fragment
  $\Rightarrow 2n$ vertices ($\sim 2n$ formulas) and central symmetry ($\sim$ contradiction)
- $\mathcal{P}$ has a symmetry group $\mathcal{S}_{\mathcal{P}}$
  - contains the reflectional and rotational symmetries of $\mathcal{P}$
  - the cardinality $|\mathcal{S}_{\mathcal{P}}|$ measures how 'symmetric' $\mathcal{P}$ is
- strictly based on the **geometrical** properties of the polygon/polyhedron
- independent of the logical structure that is being visualized
- example: a square has 8 reflectional/rotational symmetries, i.e. $|\mathcal{S}_{\mathsf{sq}}| = 8$

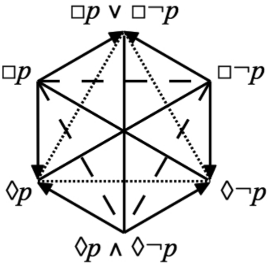


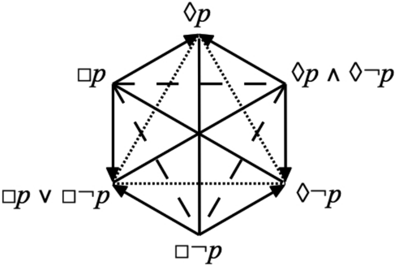
Interaction between logic and geometry

- visualize n-PCD fragment by means of $\mathcal{P}$
  - logical number: $2^n \times n!$
  - geometrical number: $|\mathcal{S}_{\mathcal{P}}|$

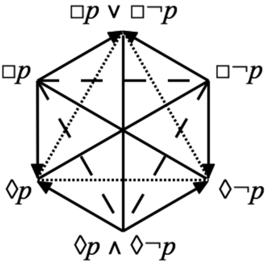
- $2^n \times n! \geq |\mathcal{S}_{\mathcal{P}}|$ (typically: $>$ instead of $\geq$)
  - every symmetry of $\mathcal{P}$ can be seen as the result of permuting/changing the orientation of the PCDs
  - but typically not vice versa

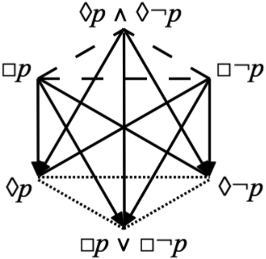
- example
  - reflect the hexagon around the axis defined by $\Box p$ and $\Diamond\neg p$
  - permute the PCDs $(\Diamond p, \Box\neg p)$ and $(\Box p \vee \Box\neg p, \Diamond p \wedge \Diamond\neg p)$





- example
  - change the orientation of the PCD $(\Box p \vee \Box\neg p, \Diamond p \wedge \Diamond\neg p)$
  - no reflectional/rotational symmetry





- work up to symmetry: $\frac{2^n \times n!}{|\mathcal{S}_\mathcal{P}|}$ fundamental forms
  - diagrams with **same fundamental form**
    $\Rightarrow$ reflectional/rotational variants of each other
  - diagrams with **different fundamental forms**:
    $\Rightarrow$ not reflectional/rotational variants of each other

- one n-PCD fragment, two different visualizations $\mathcal{P}$ and $\mathcal{P}'$

  $\mathcal{P}$ is less symmetric than $\mathcal{P}'$

  $\Leftrightarrow |\mathcal{S}_\mathcal{P}| < |\mathcal{S}_{\mathcal{P}'}|$

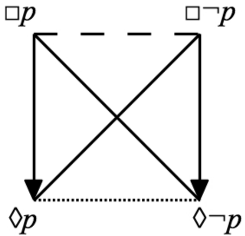
  $\Leftrightarrow \frac{2^n \times n!}{|\mathcal{S}_\mathcal{P}|} > \frac{2^n \times n!}{|\mathcal{S}_{\mathcal{P}'}|}$

  $\Leftrightarrow \mathcal{P}$ has more fundamental forms than $\mathcal{P}'$

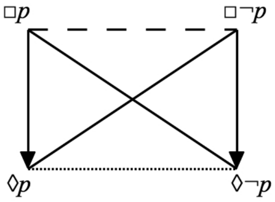
- diagrams $\mathcal{P}$ and $\mathcal{P}'$ for the same n-PCD fragment
  - $\mathcal{P}$ is less symmetric than $\mathcal{P}'$, i.e. has more fundamental forms than $\mathcal{P}'$
  - $\mathcal{P}$ makes some visual distinctions that are not made by $\mathcal{P}'$
- the diagrammatic quality of $\mathcal{P}$ and $\mathcal{P}'$ depends on whether these additional **visual distinctions** correspond to any **logical distinctions** in the underlying fragment (recall the Congruence Principle)
- if there are such logical distinctions in the fragment:
  - $\mathcal{P}$ visualizes these logical distinctions (different fundamental forms)
  - $\mathcal{P}'$ collapses these logical distinctions (same fundamental form)
  - $\mathcal{P}$ is better visualization than $\mathcal{P}'$
- if there are no such logical distinctions in the fragment:
  - no need for any visual distinctions either
  - different fundamental forms of $\mathcal{P}$: by-products of its lack of symmetry
  - $\mathcal{P}'$ is better visualization than $\mathcal{P}$

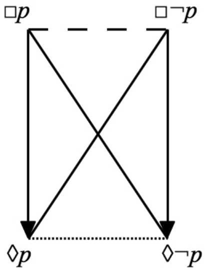
- in general: $\frac{n! \times 2^n}{|\mathcal{S}_{\mathcal{P}}|}$ fundamental forms
- 2-PCD fragment $\Rightarrow 2! \times 2^2 = 8$ configurations of PCDs
- some visualizations that have been used in the literature:
  - **square**: $|\mathcal{S}_{\mathsf{sq}}| = 8$ $\qquad \frac{2! \times 2^2}{|\mathcal{S}_{\mathsf{sq}}|} = \frac{8}{8} =$ **1 fundamental form**
  - (proper) **rectangle**: $|\mathcal{S}_{\mathsf{rect}}| = 4$ $\qquad \frac{2! \times 2^2}{|\mathcal{S}_{\mathsf{rect}}|} = \frac{8}{4} =$ **2 fundamental forms**
- Aristotelian families of 2-PCD fragments:
  - **classical**
  - degenerate

- 1 fundamental form
- no visual distinction between long and short edges (all edges are equally long)



- 2 fundamental forms
- visual distinction: long vs short edges
  - (sub)contrariety on long edges, subalternation on short edges
  - (sub)contrariety on short edges, subalternation on long edges





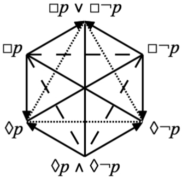
- is there a distinction between (sub)contrariety and subalternation?

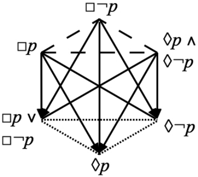
- **yes, there is**
  - complementary perspectives on the classical 'square' of opposition:
    - as a theory of negation (commentaries on *De Interpretatione*)
    - as a theory of logical consequence (commentaries on *Prior Analytics*)
  - focus on different Aristotelian relations:
    - theory of negation $\Rightarrow$ focus on (sub)contrariety
    - theory of consequence $\Rightarrow$ focus on subalternation
  - rectangle does justice to these differences (square would collapse them)

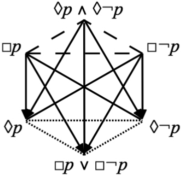
- **no, there isn't**
  - logical unity of all the Aristotelian relations
    - every (sub)contrariety yields two corresponding subalternations
    - every subalternation yields corresponding contrariety and subcontrariety
  - square does justice to this unity
    (rectangle would introduce artificial differences)

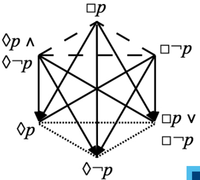
- in general: $\frac{n! \times 2^n}{|\mathcal{S}_{\mathcal{P}}|}$ fundamental forms
- 3-PCD fragment $\Rightarrow 3! \times 2^3 = 48$ configurations
- some visualizations that have been used in the literature:
  - **hexagon**: $|\mathcal{S}_{\mathsf{hex}}| = 12$ $\frac{3! \times 2^3}{|\mathcal{S}_{\mathsf{hex}}|} = \frac{48}{12} =$ **4 fundamental forms**
  - **octahedron**: $|\mathcal{S}_{\mathsf{octa}}| = 48$ $\frac{3! \times 2^3}{|\mathcal{S}_{\mathsf{octa}}|} = \frac{48}{48} =$ **1 fundamental form**
- Aristotelian families of 3-PCD fragments:
  - **Jacoby-Sesmat-Blanché (JSB)**
  - Sherwood-Czezowski (SC)
  - unconnected-4 (U4)
  - unconnected-8 (U8)
  - unconnected-12 (U12)

- 4 fundamental forms
- visual distinction:
  - all three contrariety edges equally long
  - one contrariety edge longer than the other two



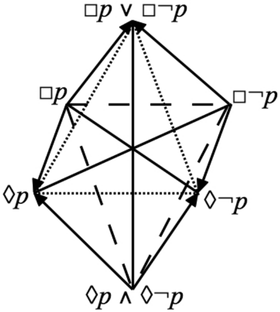






Octahedron visualization of a JSB 3-PCD fragment

- 1 fundamental form
- no visual distinction between long and short contrariety edges (all contrariety edges are equally long)



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- are there different kinds of contrariety?

- usually, the contrary formulas are modeled as elements of $\mathbb{B}_3$
  - bitstrings 100, 010 and 001
  - all contrarieties are equally 'strong'

- for linguistic/cognitive reasons, it is sometimes useful to model the contrary formulas as elements of, say, $\mathbb{B}_5$
  - bitstrings 10000, 01110, 00001
  - the contrariety 10000–00001 is 'stronger' than the two other contrarieties

- in the hexagon: edge length $\leftrightsquigarrow$ contrariety strength (Congruence)
- in the octahedron: no distinction possible (collapse)

$\Rightarrow$ **hexagon** is the **preferred visualization**

- systematic approach to informationally equivalent Aristotelian diagrams: logic (PCD structure) vs geometry (symmetry group)
- applied to some Aristotelian families of 2-PCD and 3-PCD fragments

- in general: to visualize an n-PCD fragment, consider a polytope
  - that is centrally symmetric
  - that has $2n$ vertices
  - that has a symmetry group of order $2^n \times n!$

  $\Rightarrow$ **cross-polytope of dimension** n $\Rightarrow$ 1 fundamental form
  (dual of the n-dimensional hypercube)

- diagrammatically ineffective (>3D beyond human visual cognition)
- but theoretically important: first few cases:
  - $n = 2$: 2D cross-polytope: dual of the square: **square**
  - $n = 3$: 3D cross-polytope: dual of the cube: **octahedron**

- a Hasse diagram visualizes a **partially ordered set** $(P, \leq)$:

$\leq$ is reflexive:	for all $x \in P : x \leq x$
$\leq$ is transitive:	for all $x, y, z \in P : x \leq y, y \leq z \Rightarrow x \leq z$
$\leq$ is antisymmetric:	for all $x, y \in P : x \leq y, y \leq x \Rightarrow x = y$

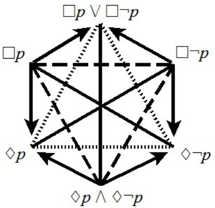
- Hasse diagrams in logic and mathematics:

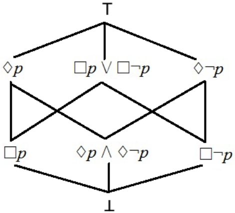
divisibility poset	$x \leq y$ iff x divides y
subgroup lattices	$x \leq y$ iff x is a subgroup of y
logic/Boolean algebra	$x \leq y$ iff x logically entails y

- we focus on **Boolean algebras**
  - always have a Hasse diagram that is **centrally symmetric**
  - can be partitioned into **levels** $L_0, L_1, L_2, \ldots, L_{n-1}, L_n$

- three key differences between Aristotelian and Hasse diagrams:
  1. the non-contingent formulas $\bot$ and $\top$
  2. the general direction of the entailments
  3. visualization of the levels

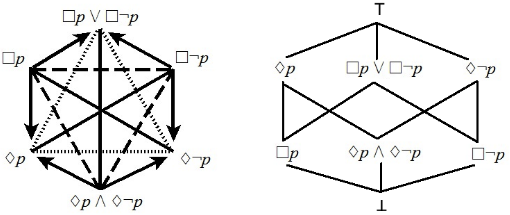
- first difference: the non-contingent formulas $\bot$ and $\top$
  - Hasse diagrams: **visualized**, as begin-/end-points of the $\leq$-ordering
  - Aristotelian diagrams: $\bot$ and $\top$ are usually **not visualized**
  - Sauriol, Smessaert, etc.: $\bot$ and $\top$ are in Aristotelian diagrams after all: $\bot$ and $\top$ coincide in the diagram's **center of symmetry**





- second difference: the general direction of the entailments
  - Hasse diagrams: all entailments go **upwards**
  - Aristotelian diagrams: **no single shared direction**

- third difference: visualization of the levels
  - Hasse diagrams: levels L_i are visualized as **horizontal hyperplanes**
  - Aristotelian diagrams: **no uniform visualization** of levels



- recall the **Congruence Principle**:
  - the content/structure of the visualization corresponds to the content/structure of the desired mental representation
  - cf. Barbara Tversky et al.

- different visual properties $\leftrightsquigarrow$ different goals
  - Aristotelian diagrams: visualize the **Aristotelian relations**
  - Hasse diagrams: visualize the structure of the **entailment ordering** $\leq$

- Hasse diagrams: strong congruence between logical and visual properties
  - **shared direction of entailment** (vertically upward)
  - **levels** as horizontal lines/planes
    - if $\varphi, \psi \in L_i$, then $\varphi \nleq \psi$ and $\psi \nleq \varphi$
    - formulas of a single level are **independent** of each other w.r.t. $\leq$
    - level = **horizontal** $\Rightarrow$ **orthogonal** to the **vertical** $\leq$-direction

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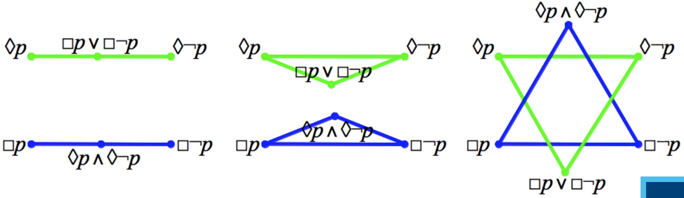
- consider the three S5-formulas $\Box p$, $\Box\neg p$, $\Diamond p \wedge \Diamond\neg p$
  - Hasse perspective: all belong to $L_1 \Rightarrow$ horizontal line
  - Aristotelian perspective: all contrary to each other
- the contrariety between $\Box p$ and $\Box\neg p$ overlaps with the two others
  - serious violation of the **Apprehension Principle**
  - direct reason: the three formulas lie on a single line
- this is solved in the Aristotelian diagram:
  - move $\Diamond p \wedge \Diamond\neg p$ away from the line between $\Box p$ and $\Box\neg p$
  - triangle of contrarieties $\Rightarrow$ in line with Apprehension Principle
  - mixing of levels, no single entailment direction, $\perp$ moves to middle



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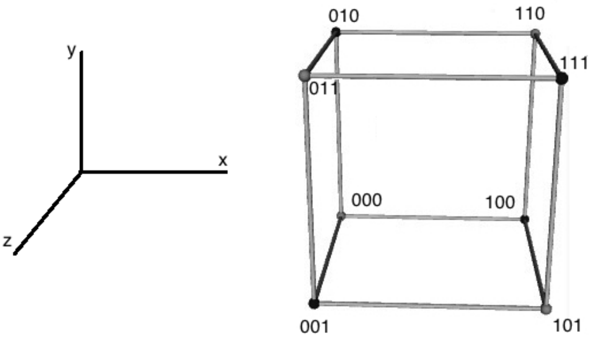
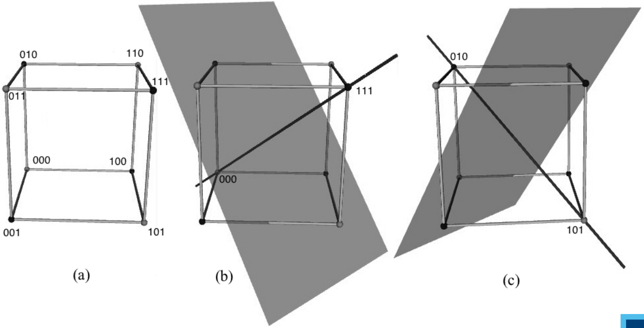
- we restrict ourselves to Aristotelian diagrams that are Boolean closed
- the Hasse diagram of $\mathbb{B}_3$ can be drawn as a three-dimensional **cube**
  - general entailment direction runs from 000 to 111
  - logical levels $\leftrightsquigarrow$ planes orthogonal to the entailment direction



- in (a) the cube consists of 4 pairs of diametrically opposed vertices:
  - 3 contingent pairs: 101—010, 110—001, 011—100
  - 1 non-contingent pair: 000—111
- each pair defines a projection axis for **vertex-first parallel projection**:
  - in (b) projection along 000—111 axis
  - in (c) projection along 101—010 axis



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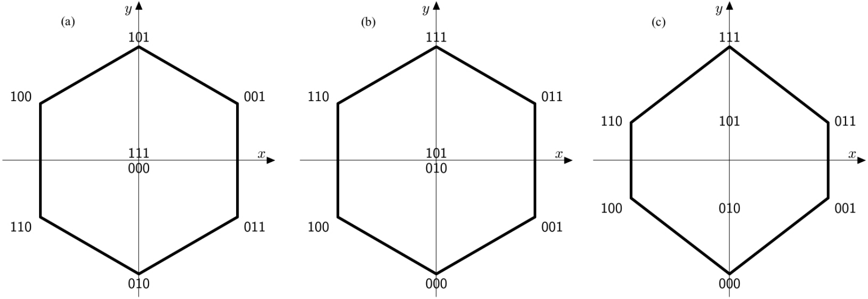
- the vertex-first projections from 3D cube to 2D hexagon:
  (a) projection along 000—111 $\Rightarrow$ **Aristotelian diagram** (JSB hexagon)
  (b) projection along 101—010 $\Rightarrow$ Hasse diagram (almost)
- if we slightly 'nudge' the projection axis 101—010, we get:
  (c) projection 'along' 101—010 $\Rightarrow$ **Hasse diagram**

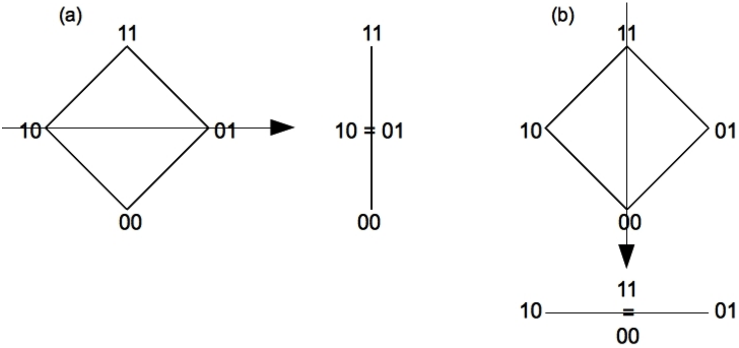


- both Aristotelian and Hasse diagram are **vertex-first parallel projections of cube**:
  - Aristotelian diagram: project along the **entailment direction** (000—111)
  - Hasse diagram: project along **another direction** (101—010)
- recall the dissimilarities between Aristotelian and Hasse diagrams:
  1. the position of $\perp$ and $\top$
  2. the general direction of the entailments
  3. the visualization of the levels
- these three differences turn out to be interrelated: different manifestations of a single choice (**projection axis**)
- now: illustrate these differences by means of a more basic vertex-first projection (from 2D square to 1D line)

difference 1: the position of $\perp$ and $\top$

the square is a Hasse diagram $\Rightarrow$ $\perp$ and $\top$ as lowest and highest point

(a) project along other direction $\Rightarrow$ $\perp$ and $\top$ still as **lowest and highest**
(b) project along the $\top/\perp$ direction $\Rightarrow$ $\perp$ and $\top$ **coincide in the center**



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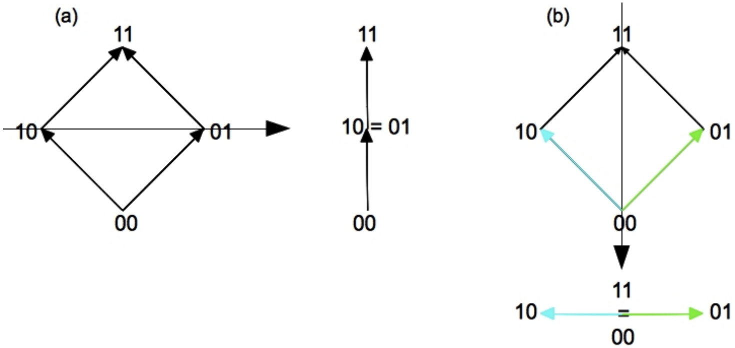
difference 2: the general direction of the entailments

the square is a Hasse diagram ⇒ general entailment direction is upwards

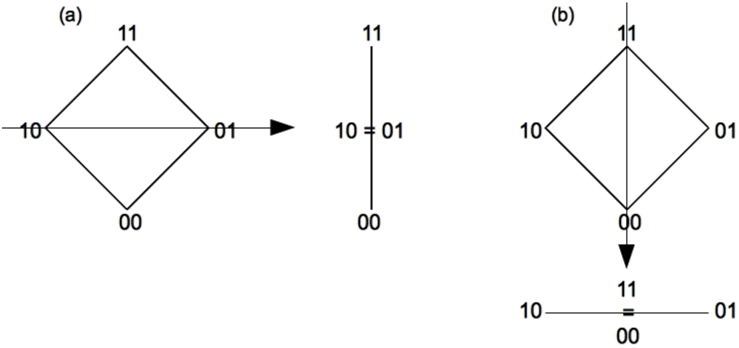
(a) project along other direction ⇒ general entailment direction is still **upwards**
(b) project along the ⊤/⊥ direction ⇒ **no general entailment direction** anymore



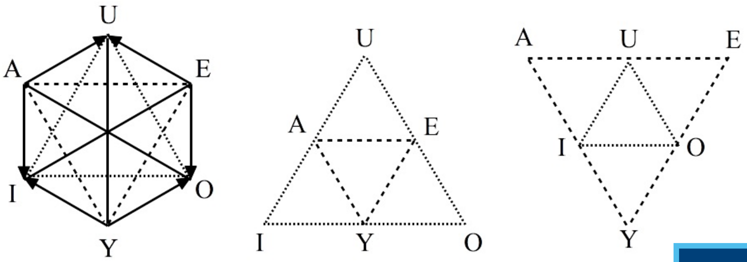
difference 3: the visualization of the levels

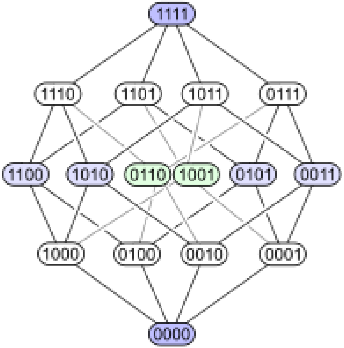
the square is a Hasse diagram $\Rightarrow$ uniform (horizontal) levels

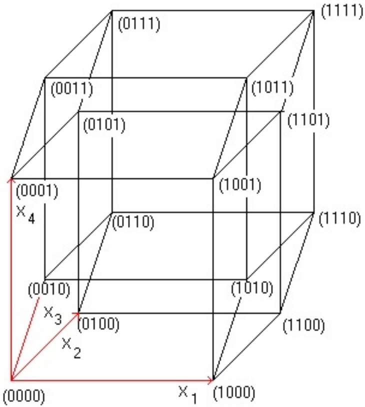
(a) project along other direction $\Rightarrow$ still **uniform** (horizontal) levels
(b) project along the $\top/\bot$ direction $\Rightarrow$ **mixing** of levels



- vertex-first projection along the $\top/\bot$ direction (mixing of levels)
- this can be a **parallel** projection
  - from cube to **hexagon**
  - interlocking **same-sized** triangles for contrariety and subcontrariety
- this can be a **perspective** projection
  - from cube to **'nested triangles'**
  - nested **different-sized** triangles for contrariety and subcontrariety




1111
1110
1101
1011
0111
1100
1010
0110
1001
0101
0011
1000
0100
0010
0001
0000

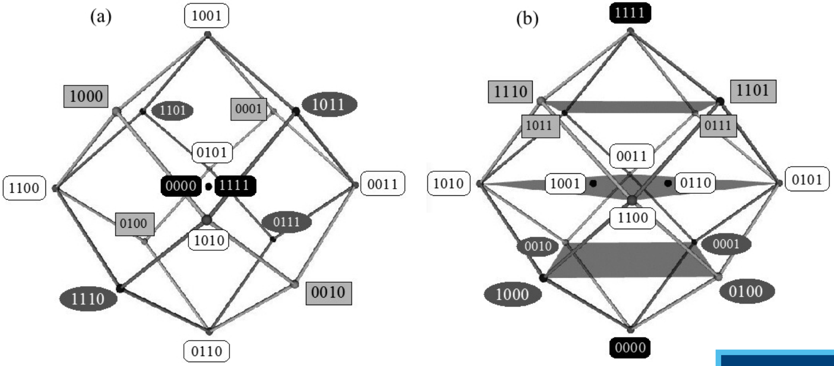

(1111)
(0111)
(0011)
(1011)
(0101)
(1101)
(0001)
(1001)
X4
(0110)
(1110)
(0010)
(1010)
X3
(0100)
(1100)
X2
(0000)
X1
(1000)

KU LEUVEN

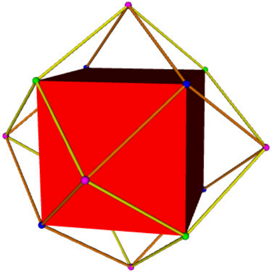
Modal Logic S5	Propositional Logic	bitstrings level 1	bitstrings level 3	Propositional Logic	Modal Logic S5
$\Box p$	$p \wedge q$	1000	0111	$\neg(p \wedge q)$	$\neg\Box p$
$\neg\Box p \wedge p$	$\neg(p \rightarrow q)$	0100	1011	$p \rightarrow q$	$\Box p \vee \neg p$
$\Diamond p \wedge \neg p$	$\neg(p \leftarrow q)$	0010	1101	$p \leftarrow q$	$\neg\Diamond p \vee p$
$\neg\Diamond p$	$\neg(p \vee q)$	0001	1110	$p \vee q$	$\Diamond p$

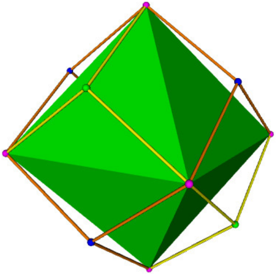
Modal Logic S5	Propositional Logic	bitstrings level 2/0	bitstrings level 2/4	Propositional Logic	Modal Logic S5
p	p	1100	0011	$\neg p$	$\neg p$
$\Box p \vee (\Diamond p \wedge \neg p)$	q	1010	0101	$\neg q$	$\neg\Diamond p \vee (\neg\Box p \wedge p)$
$\Box p \vee \neg\Diamond p$	$p \leftrightarrow q$	1001	0110	$\neg(p \leftrightarrow q)$	$\neg\Box p \wedge \Diamond p$
$\Box p \wedge \neg\Box p$	$p \wedge \neg p$	0000	1111	$p \vee \neg p$	$\Box p \vee \neg\Box p$

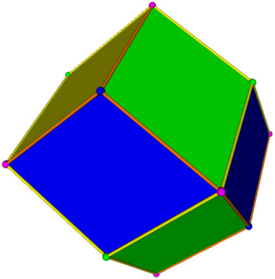
- **vertex-first parallel projection**
- from 4D hypercube to 3D **rhombic dodecahedron** (RDH)
  - along the 0000—1111 axis ⇒ Aristotelian RDH (Smessaert, Demey)
  - along the 1001—0110 axis ⇒ Hasse RDH (Zellweger)



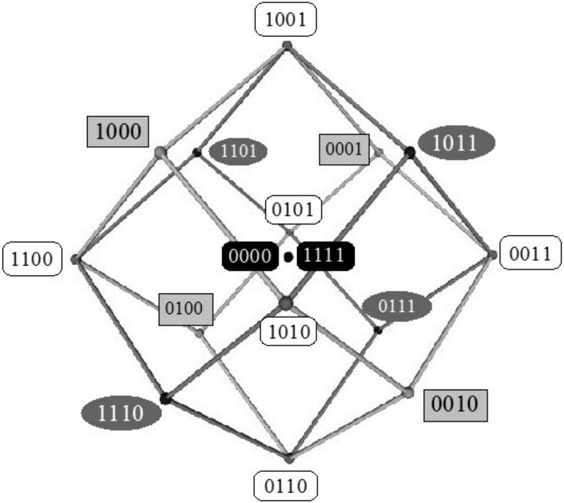
cube	+	octahedron	=	cuboctahedron	$\stackrel{dual}{\Longrightarrow}$	**rhombic dodecahedron**
Platonic		Platonic		Archimedean		Catalan
6 faces 8 vertices 12 edges		8 faces 6 vertices 12 edges		14 faces 12 vertices 24 edges		**12 faces** **14 vertices** **24 edges**

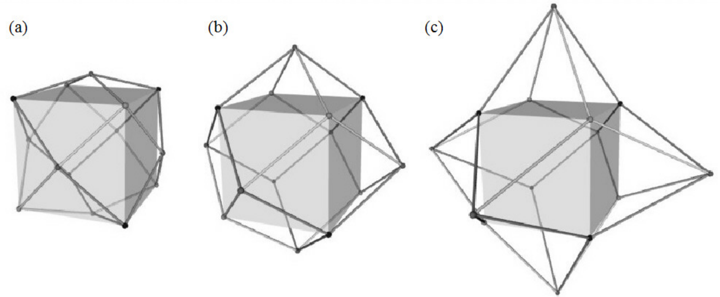




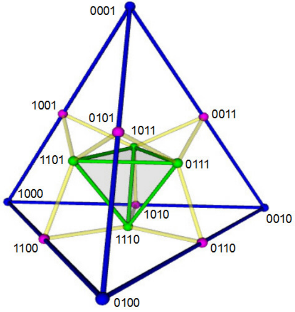


cube: $4 \times$ L1 $+ 4 \times$ L3 / **octahedron**: $6 \times$ L2 / **center**: $1 \times$ L0 $+ 1\times$ L4





tetra(kis)-hexahedron	**rhombic dodecahedron**	**tetra-icosahedron**
THH	RDH	TIH
(Sauriol/Pellissier)	(Smessaert/Demey)	(Moretti)
14 vertices	14 vertices	14 vertices
24 faces/36 edges	12 faces/24 edges	24 faces/36 edges
convex	convex	non-convex

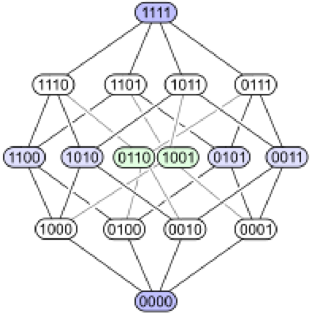


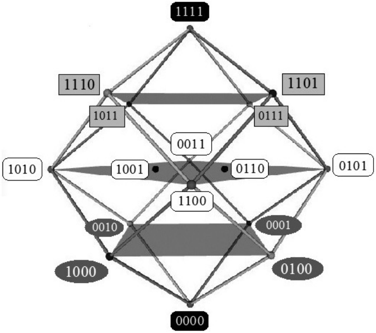
nested tetrahedron (NTH)

(Dubois & Prade, Ciucci, Lewis Carroll, Moretti)

4 faces + 4 vertices + 6 edges

vertex-first **perspective projection**
of a 4D hypercube along the 0000—1111 axis

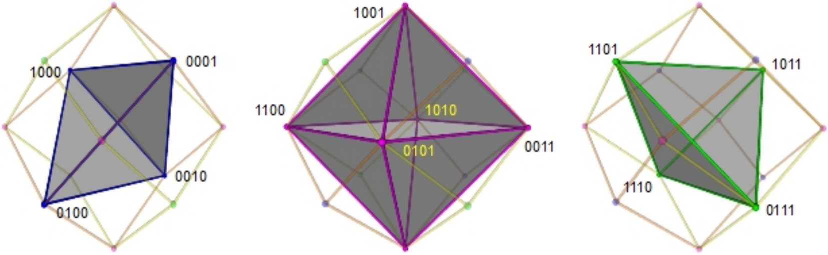




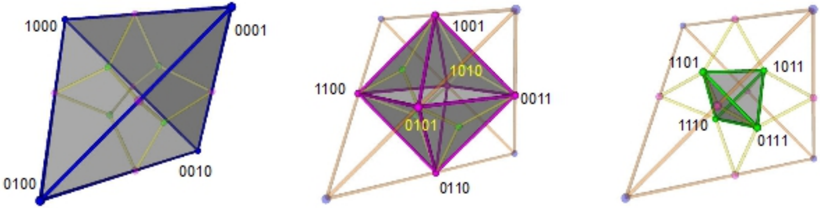
logical levels are geometrically represented as horizontal planes orthogonal to the vertical implication direction

Congruence Principle

structure of visualization $\sim$ represented logical structure



levels are **not** parallel planes
levels are **not** geometrical dimensions
$\Downarrow$
Aristotelian RDH is **not** level-preserving
(violating the Congruence Principle)



levels are not parallel planes, but are **geometrical dimensions**

L1	$\sim$	zero-dimensionality	$\rightsquigarrow$	4 **vertices**
L2	$\sim$	one-dimensionality	$\rightsquigarrow$	midpoints of 6 **edges**
L3	$\sim$	two-dimensionality	$\rightsquigarrow$	midpoints of 4 **faces**

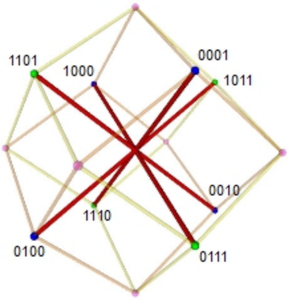
$\Downarrow$

NTH is level-preserving
(observing the Congruence Principle)

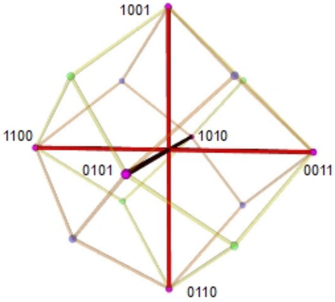
- the contradiction relation is symmetric and functional
  - Aristotelian diagrams (usually) represent *CD* by **central symmetry**
  - contradictory bitstrings are located at diametrically opposed vertices at the same distance from the diagram's centre

- Congruence Principle: logical distance $\sim$ geometrical distance:
  - **Hamming distance**: $d_H(b, b') :=$ number of bit values switched
  - **Euclidean distance**: $d_{RDH}(b, b') := d_E(c_{RDH}(b), c_{RDH}(b'))$
  - $c_{RDH}(b) :=$ Euclidean coordinates of the vertex representing b in RDH
  - $d_H(b_1, b_2) < d_H(b_3, b_4) \implies d_{RDH}(b_1, b_2) < d_{RDH}(b_3, b_4)$

- contradiction relation = **strongest** opposition relation
  - contradiction = switching all bit values = maximal Hamming distance
  - congruence: **maximal** logical distance $\sim$ **maximal** geometrical distance
  - $c_{RDH}(b)$ is farthest removed from $c_{RDH}(\neg b)$
  - $\arg\max_{x \in \mathbb{B}_4} d_H(b, x) = \neg b = \arg\max_{x \in \mathbb{B}_4} d_{RDH}(b, x)$



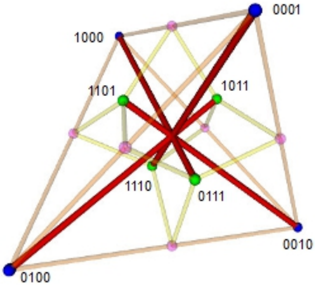
L1-L3 contradiction
central symmetry
maximal distance

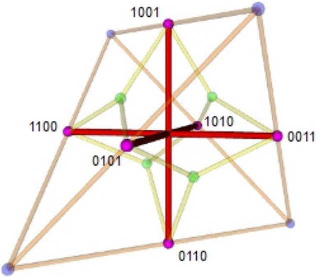


L2-L2 contradiction
central symmetry
maximal distance

⇓

RDH observes the Congruence Principle



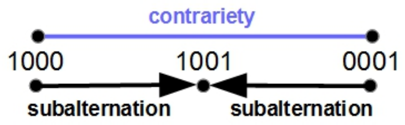


L1-L3 contradiction
no central symmetry
no maximal distance

L2-L2 contradiction
central symmetry
no maximal distance

⇓

NTH violates the Congruence Principle

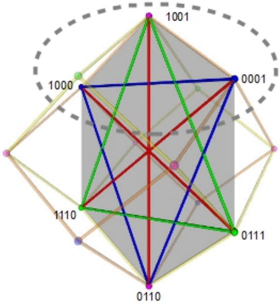


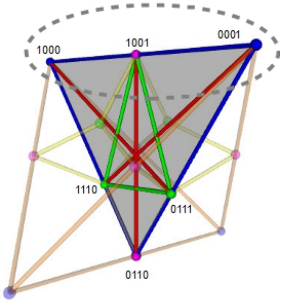
3 **distinct** logical relations (opposition/implication)

versus

3 **distinct/coinciding** visual components (line/arrow)

Apprehension Principle:
the content/structure of the visualisation
can readily and correctly be perceived and understood





no triples of collinear vertices
no visual overlap/coincidence
⇓
RDH observes Apprehension

triples of collinear vertices
visual overlap/coincidence
⇓
NTH violates Apprehension

Thank you! Questions?

More info: www.logicalgeometry.org