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Introduction to Logical Geometry

1. Basic Concepts and Bitstring Semantics

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ESSLLI 2018, Sofia

- lecturers:
  - Lorenz Demey
    - primary background in logic and philosophy
    - Center for Logic and Philosophy of Science, KU Leuven
    - `http://www.lorenzdemey.eu`
  - Hans Smessaert
    - primary background in linguistics
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    - `http://wwwling.arts.kuleuven.be/ComForT/hsmessaert/`
- course website:
  - `http://logicalgeometry.org/tutorial-esslli2018.htm`
  - course slides
  - background readings

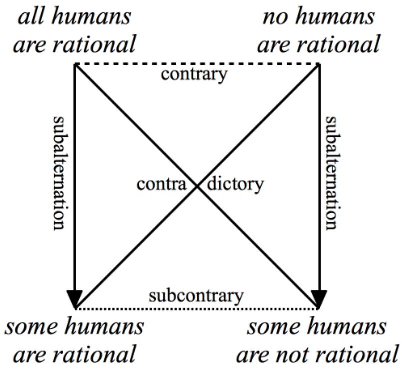
Who are you?

- what's your academic background?
  - philosophy
  - logic
  - linguistics
  - mathematics
  - computer science

  ⋮

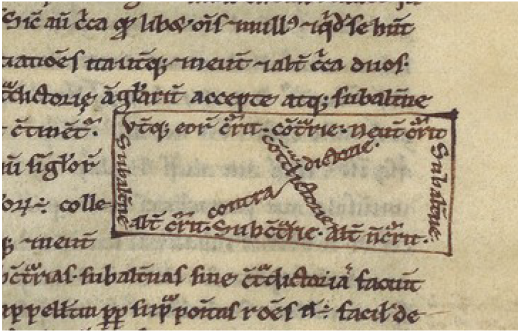
- logical geometry $\sim$ the systematic study of Aristotelian diagrams
- what are Aristotelian diagrams?
  - later: precise definition
  - now: some motivating examples
- some general trends to pay attention to:
  - long history, but still used today
  - applications in logic and philosophy, but also in many other disciplines
  - not just for teaching purposes, but also in research contexts

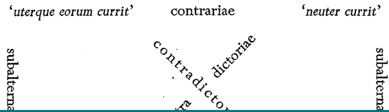
- oldest and most well-known example of an Aristotelian diagram
- the square of opposition for the categorical statements from syllogistics
  - relations: Aristotle (4th century BCE)
  - diagram: Apuleius of Madaura (2nd century CE),
    Boethius (5th century CE)



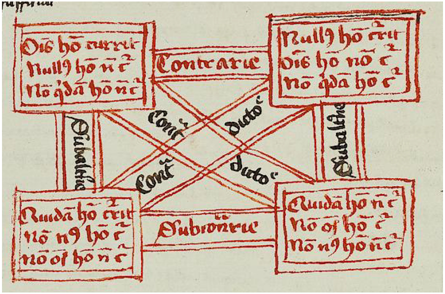
- epic poem: *Der Wälsche Gast*
- visual representation of the seven liberal arts

- square for the quantifiers from the categorical statements (all, some, no)
- also a square for the dual quantifiers (both, either, neither)





- squares for the quantifiers and the modalities
- within each vertex: duality behavior
  - **every** man runs
  - **no** man does **not** run
  - **not some** man does **not** run



	CONTRARIE	
Non possibile est non esse	Tertius est quarto semper contrarius ordo	Non possibile est esse
Non contingens est non esse		Non contingens est esse
Impossibile est non esse		Impossibile est esse
Necesse est esse		Necesse est non esse

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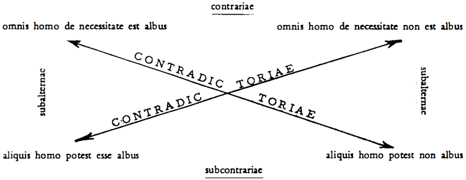
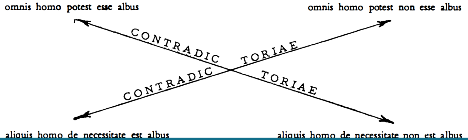
- modal syllogistics: propositions with quantifiers and modalities
- ‘figura completa’, but also ‘figura incompleta’

Sic igitur per istas propositiones habetur una figura completa habens propositiones contrarias, contradictorias, subalternas et subcontrarias sic dispositas:

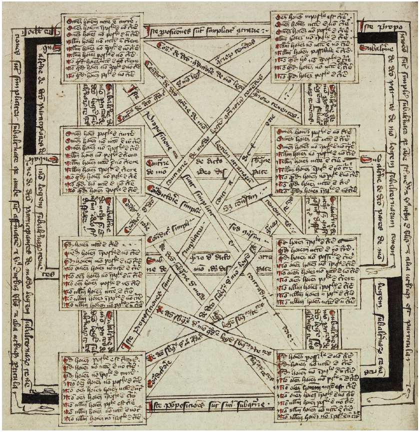


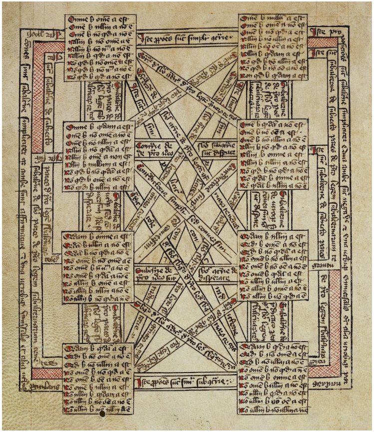
contrariantur, quia possunt esse simul falsae. Et ita habetur tertia figura, sed incompleta, talis:



John Buridan (1300 – 1358)

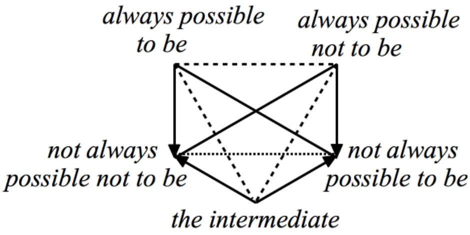
- integrates several squares into one 'magna figura'
- for modal syllogistics, but also for other types of propositions

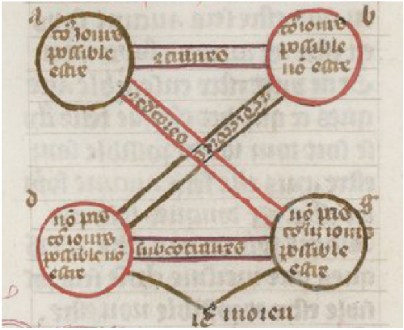




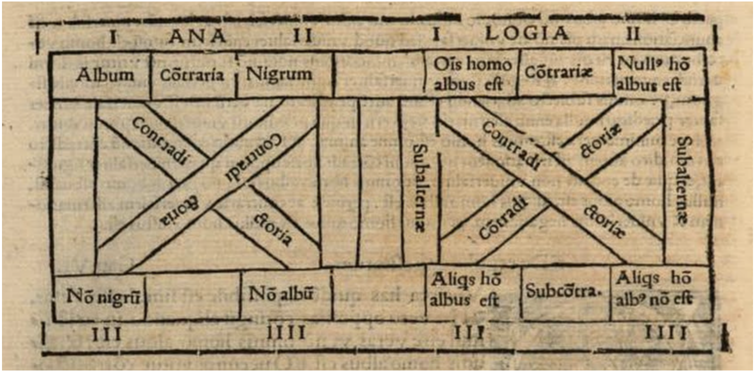
Nicole Oresme (1323 – 1382)

- (proto-scientific) cosmology: *Livre du Ciel et du Monde*
- an ‘extended’ square: add the conjunction of the two lower corners





- analogy:
  - a square for propositions // a square for properties
  - 'cannot be true together' // 'cannot be instantiated together'



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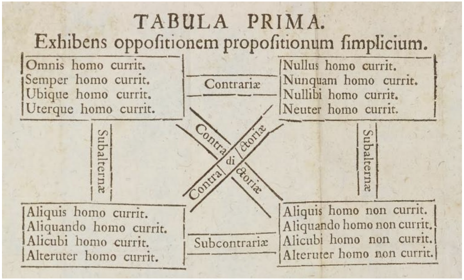
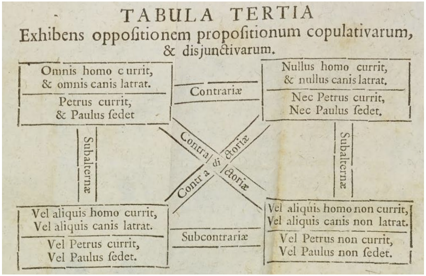
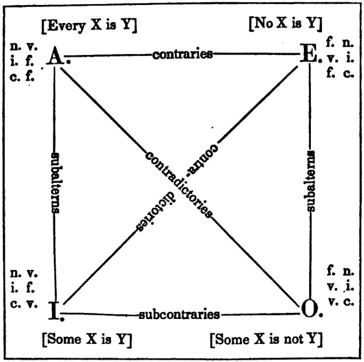
- squares for quantifiers, propositional connectives, modalities, temporal and spatial adverbs





- "In the nineteenth century, the apparently most widely used textbook in Britain and America" (Parsons, 2017)
- usual square for the categorical statements
- three types of matter (connection between subject and predicate): [n]ecessary, [i]mpossible and [c]ontingent



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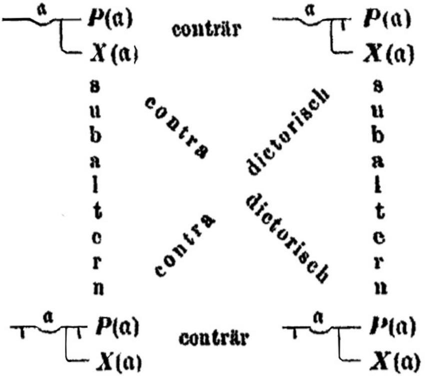
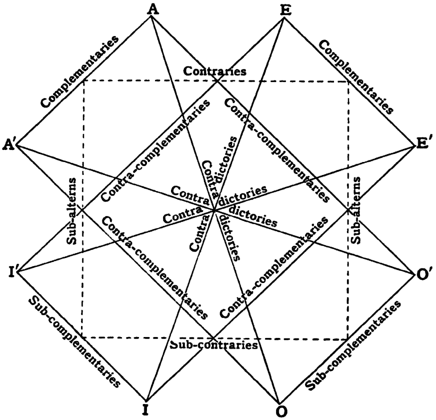
Gottlob Frege (1848 – 1925)

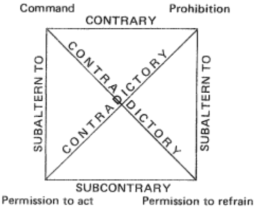
- a square of opposition in Begriffschrift notation
- note the mistake: 'conträr' $\rightsquigarrow$ 'subconträr'



- octagon for the categorical statements with subject negation

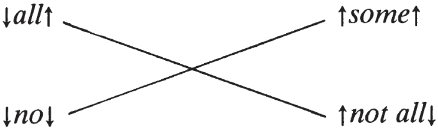


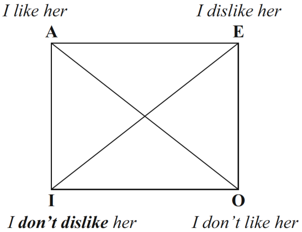
- Ruth Barcan Marcus
- Arthur Prior
- Hans Reichenbach
- Richard Hare
- H. L. A. Hart (cf. figure)
- Roderick Chisholm
- Ernest Sosa



Applications beyond logic and philosophy

- linguistics
  - semantics (generalized quantifiers) (Dag Westerståhl)
  - pragmatics (implicatures) (Laurence Horn)
  - typology (lexicalization) (Debra Ziegeler)





On the empty O-corner of the Aristotelian Square:
A view from Singapore English

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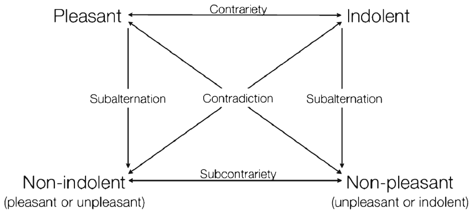
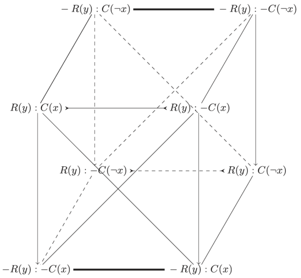
- cognitive science
  - psychology of reasoning (Stephen Newstead, Richard Griggs)
  - emotions research (Olivier Massin)
  - neuroscience (Camillo Porcaro et al.)

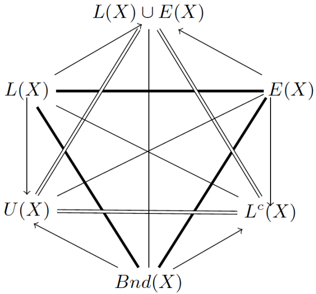


Drawing Inferences from Quantified Statements: A Study of the Square of Opposition

Universal vs. particular reasoning: a study with neuroimaging techniques

- computer science (knowledge representation)
  - formal concept analysis (Didier Dubois, Henri Prade)
  - rough set theory (Yiyu Yao, Davide Ciucci)
  - formal argumentation theory (Leila Amgoud)





The Cube of Opposition -
A Structure underlying many Knowledge Representation Formalisms

- Aristotelian diagrams have been used
  - for a very long time (including today)
  - in a wide variety of disciplines (not just logic and philosophy)

- Aristotelian diagrams constitute a **language** for a broad (transdisciplinary and transhistorical) community of researchers who deal with logical reasoning

- logical geometry $\sim$ the **linguistics** that systematically studies the language of Aristotelian diagrams

- two fundamental aspects of any language:
  - **syntax**: form, representation $\rightsquigarrow$ 'geometry'
  - **semantics**: meaning, what is represented $\rightsquigarrow$ 'logical'

- **perspective shift**:
  - in a typical application:
    Aristotelian diagrams are used (= tool)
    to analyze some linguistic, logical, conceptual phenomenon (= object)
  - in logical geometry:
    Aristotelian diagrams are themselves the primary objects of study,
    analyzed using a variety of tools (bitstring analysis, group theory, etc.)

- this has led to an elaborate (and growing) **elegant theory**
  (regardless of the multitude of applications)

- **double motivation** for logical geometry:
  - Aristotelian diagrams as objects of independent interest
  - Aristotelian diagrams as a widely-used language

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- other types of logic diagrams:
  - Hasse diagrams
  - Euler/Venn diagrams
  - duality diagrams

- since the 1990s: diagrammatic reasoning

- two courses at ESSLLI 2017:
  - Caught in the Spiders' Diagrammatic Reasoning Web – The Euler/Spider Diagram Family of Formal Reasoning Systems
  - Picturing Quantum Processes

We provide a self-contained introduction to quantum theory . . . This course is unique in our use of a diagrammatic language throughout. Far from simple visual aids, the diagrams we use are mathematical objects in their own right

Structure of the course

1. Basic Concepts and Bitstring Semantics
2. Abstract-Logical Properties of Aristotelian Diagrams, Part I
   ☞ Aristotelian, Opposition, Implication and Duality Relations
3. Visual-Geometric Properties of Aristotelian Diagrams
   ☞ Informational Equivalence, Symmetry and Distance
4. Abstract-Logical Properties of Aristotelian Diagrams, Part II
   ☞ Boolean Structure and Logic-Sensitivity
5. Case Studies and Philosophical Outlook

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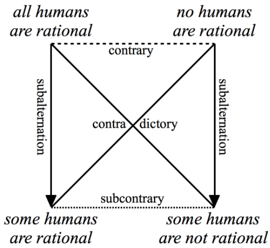
5. Case Studies and Philosophical Outlook

- two propositions are said to be

contradictory (CD)	iff	they cannot be true together and they cannot be false together
contrary (C)	iff	they cannot be true together but they can be false together
subcontrary (SC)	iff	they can be true together but they cannot be false together
in subalternation (SA)	iff	the first proposition entails the second but the second doesn't entail the first



- let S be a **logical system** with
  - the usual Boolean connectives $(\wedge, \vee, \neg, \rightarrow)$
  - a model-theoretic semantics $(\models)$

- two formulas $\varphi, \psi \in \mathcal{L}_\mathsf{S}$ are said to be

S-*contradictory* (CD_S)	iff	$\models_\mathsf{S} \neg(\varphi \wedge \psi)$	and	$\models_\mathsf{S} \neg(\neg\varphi \wedge \neg\psi)$
S-*contrary* (C_S)	iff	$\models_\mathsf{S} \neg(\varphi \wedge \psi)$	and	$\not\models_\mathsf{S} \neg(\neg\varphi \wedge \neg\psi)$
S-*subcontrary* (SC_S)	iff	$\not\models_\mathsf{S} \neg(\varphi \wedge \psi)$	and	$\models_\mathsf{S} \neg(\neg\varphi \wedge \neg\psi)$
in S-*subalternation* (SA_S)	iff	$\models_\mathsf{S} \varphi \rightarrow \psi$	and	$\not\models_\mathsf{S} \psi \rightarrow \varphi$

- the Aristotelian geometry for S: $\mathcal{AG}_\mathsf{S} := \{CD_\mathsf{S}, C_\mathsf{S}, SC_\mathsf{S}, SA_\mathsf{S}\}$

- the Aristotelian relations are defined up to logical equivalence:
  - suppose that $\varphi \equiv_\mathsf{S} \varphi'$ and $\psi \equiv_\mathsf{S} \psi'$
  - then for all $R \in \mathcal{AG}_\mathsf{S}$: $R_\mathsf{S}(\varphi, \psi) \Leftrightarrow R_\mathsf{S}(\varphi', \psi')$

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- let $\mathbb{B} = \langle B, \wedge, \vee, \neg, \top, \bot \rangle$ be an arbitrary **Boolean algebra**
- two elements $x, y \in B$ are said to be

$\mathbb{B}$-*contradictory* ($CD_{\mathbb{B}}$)	iff	$x \wedge y = \bot$	and	$x \vee y = \top$
$\mathbb{B}$-*contrary* ($C_{\mathbb{B}}$)	iff	$x \wedge y = \bot$	and	$x \vee y \neq \top$
$\mathbb{B}$-*subcontrary* ($SC_{\mathbb{B}}$)	iff	$x \wedge y \neq \bot$	and	$x \vee y = \top$
in $\mathbb{B}$-*subalternation* ($SA_{\mathbb{B}}$)	iff	$\neg x \vee y = \top$	and	$x \vee \neg y \neq \top$

- the Aristotelian geometry for $\mathbb{B}$: $\mathcal{AG}_{\mathbb{B}} := \{CD_{\mathbb{B}}, C_{\mathbb{B}}, SC_{\mathbb{B}}, SA_{\mathbb{B}}\}$
- thanks to this abstract characterisation, Aristotelian relations can be defined between **formulas/statements** and between **sets/concepts**
  - cf. Lefèvre d'Étaples's 'analogia' between two squares of oppositions
  - Keynes, 1906: "These seven possible relations between *propositions* (taken in pairs) will be found to be precisely analogous to the seven possible relations between *classes* (taken in pairs)"

- first concrete instance of the algebraic characterisation: Aristotelian relations in a **Lindenbaum-Tarski algebra**
- S-equivalence classes of formulas: $[\varphi]_S := \{\psi \in \mathcal{L}_S \mid \varphi \equiv_S \psi\}$
- let $\mathbb{B}(S)$ be the Lindenbaum-Tarski algebra of the logical system S
- two equivalence classes $[\varphi]_S$, $[\psi]_S$ are said to be

$\mathbb{B}(S)$-*contradictory*	iff	$[\varphi]_S \wedge [\psi]_S = \bot$	and	$[\varphi]_S \vee [\psi]_S = \top$
$\mathbb{B}(S)$-*contrary*	iff	$[\varphi]_S \wedge [\psi]_S = \bot$	and	$[\varphi]_S \vee [\psi]_S \neq \top$
$\mathbb{B}(S)$-*subcontrary*	iff	$[\varphi]_S \wedge [\psi]_S \neq \bot$	and	$[\varphi]_S \vee [\psi]_S = \top$
in $\mathbb{B}(S)$-*subalternation*	iff	$[\neg\varphi]_S \vee [\psi]_S = \top$	and	$[\varphi]_S \vee [\neg\psi]_S \neq \top$

- this characterisation essentially corresponds to the model-theoretic one: e.g. φ and ψ are S-contrary iff $[\varphi]_S$ and $[\psi]_S$ are $\mathbb{B}(S)$-contrary

- second concrete instance of the algebraic characterisation: Aristotelian relations in a Boolean algebra of **sets**

- let $\mathbb{B} = \langle B, \cap, \cup, \setminus, D, \emptyset \rangle$ be a Boolean algebra of sets

- two sets $X, Y \in B$ are said to be

$\mathbb{B}$-*contradictory*	iff	$X \cap Y = \emptyset$	and	$X \cup Y = D$
$\mathbb{B}$-*contrary*	iff	$X \cap Y = \emptyset$	and	$X \cup Y \neq D$
$\mathbb{B}$-*subcontrary*	iff	$X \cap Y \neq \emptyset$	and	$X \cup Y = D$
in $\mathbb{B}$-*subalternation*	iff	$(D \setminus X) \cup Y = D$	and	$X \cup (D \setminus Y) \neq D$

- informal characterisation:
  two propositions φ, ψ are said to be **unconnected** iff

  (i) φ and ψ can be true together and
  (ii) φ does not entail ψ and
  (iii) ψ does not entail φ and
  (iv) φ and ψ can be false together

- together, these four conditions imply that φ and ψ do **not stand in any Aristotelian relation**:
  - condition (i) implies that φ and ψ are neither *CD* nor *C*
  - condition (ii) implies that there is no *SA* from φ to ψ
  - condition (iii) implies that there is no *SA* from ψ to φ
  - condition (iv) implies that φ and ψ are neither *CD* nor *SC*

- model-theoretic characterisation:
  two formulas φ, ψ are said to be S-**unconnected** iff

  (i) $\not\models_S \neg(\varphi \wedge \psi)$ and
  (ii) $\not\models_S \varphi \to \psi$ and
  (iii) $\not\models_S \psi \to \varphi$ and
  (iv) $\not\models_S \neg(\neg\varphi \wedge \neg\psi)$

- algebraic characterisation:
  two elements $x, y \in B$ are said to be $\mathbb{B}$-**unconnected** iff

  (i) $x \wedge y \neq \bot$ and
  (ii) $x \wedge \neg y \neq \bot$ and
  (iii) $\neg x \wedge y \neq \bot$ and
  (iv) $\neg x \wedge \neg y \neq \bot$

- first concrete instance: Lindenbaum-Tarski algebra:
  two equivalence classes $[\varphi]_S$, $[\psi]_S$ are said to be $\mathbb{B}(\mathsf{S})$-**unconnected** iff

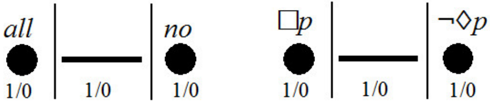
$$
\begin{array}{lrl}
\text{(i)} & [\varphi]_S \wedge [\psi]_S \neq \bot & \text{and} \\
\text{(ii)} & [\varphi]_S \wedge [\neg\psi]_S \neq \bot & \text{and} \\
\text{(iii)} & [\neg\varphi]_S \wedge [\psi]_S \neq \bot & \text{and} \\
\text{(iv)} & [\neg\varphi]_S \wedge [\neg\psi]_S \neq \bot &
\end{array}
$$

- second concrete instance: Boolean algebra of sets:
  two sets $X, Y \in B$ are said to be $\mathbb{B}$-**unconnected** iff

$$
\begin{array}{lrl}
\text{(i)} & X \cap Y \neq \emptyset & \text{and} \\
\text{(ii)} & X \cap (D \backslash Y) \neq \emptyset & \text{and} \\
\text{(iii)} & (D \backslash X) \cap Y \neq \emptyset & \text{and} \\
\text{(iv)} & (D \backslash X) \cap (D \backslash Y) \neq \emptyset &
\end{array}
$$

- bitstrings are finite **sequences of bits** (0/1), e.g. 10101011
- bitstrings can encode the denotations of formulas or expressions from:
  - **logical systems**: e.g. classical propositional logic, first-order logic, modal logic and public announcement logic
  - **lexical fields**: e.g. comparative quantification, subjective quantification, color terms and set inclusion relations
- each bit provides an **answer** to a meaningful (binary) **question** (origin: analysis of generalized quantifiers as sets of sets)
- note:
  - we use bitstrings to encode **formulas**, not **relations** between formulas
  - if a formula φ is encoded by the bitstring b, we write $\beta(\varphi) = b$
  - $[b]_i$ denotes the i^{th} bit position of the bitstring b

- each question concerns a component (point/interval) of a **scalar structure** that creates a **partition** of logical space



- application to FOL/GQT: is $Q(A, B)$ true if

$A \subseteq B$?	yes/no
$A \not\subseteq B$ and $A \cap B \neq \emptyset$?	yes/no
$A \cap B = \emptyset$?	yes/no

- examples:

β(*all A are B*) $= 100$ $= \langle$ yes, no, no $\rangle$

β(*some but not all A are B*) $= 010$ $= \langle$ no, yes, no $\rangle$

β(*not all A are B*) $= 011$ $= \langle$ no, yes, yes $\rangle$

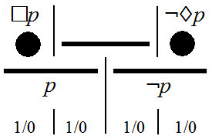
- application to the modal logic S5: is φ true if

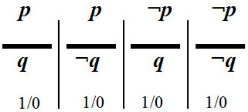
p is true in all possible worlds?	yes/no
p is true in some but not in all possible worlds?	yes/no
p is true in no possible worlds?	yes/no

- examples:

$$\beta(\Diamond p) = 110 = \langle \text{ yes, yes, no } \rangle$$
$$\beta(\Diamond p \wedge \Diamond\neg p) = 010 = \langle \text{ no, yes, no } \rangle$$
$$\beta(\Diamond\neg p) = 011 = \langle \text{ no, yes, yes } \rangle$$

Modal Logic	GQT	level 1/0	level 2/3	GQT	Modal Logic
necessary ($\Box p$)	*all*	100	011	*not all*	*not necessary* ($\neg\Box p$)
contingent ($\neg\Box p \wedge \Diamond p$)	*some but not all*	010	101	*no or all*	*not contingent* ($\Box p \vee \neg\Diamond p$)
impossible ($\neg\Diamond p$)	*no*	001	110	*some*	*possible* ($\Diamond p$)
contradiction ($\Box p \wedge \neg\Box p$)	*some and no*	000	111	*some or no*	*tautology* ($\Box p \vee \neg\Box p$)



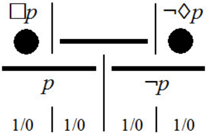


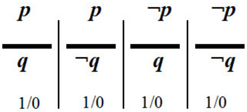
- **second** application to the modal logic S5: is φ true if

p is true in all possible worlds?	yes/no
p is true in the actual world but not in all possible worlds?	yes/no
p is true in some possible worlds but not in the actual world?	yes/no
p is true in no possible worlds?	yes/no

- examples:

$$\begin{array}{lll} \beta(\Diamond p) & = 1110 & = \langle \text{ yes, yes, yes, no } \rangle \\ \beta(\Diamond p \wedge \Diamond\neg p) & = 0110 & = \langle \text{ no, yes, yes, no } \rangle \\ \beta(\Diamond\neg p) & = 0111 & = \langle \text{ no, yes, yes, yes } \rangle \end{array}$$





- application to propositional logic: is φ true if

p is true and q is true?	yes/no
p is true and q is false?	yes/no
p is false and q is true?	yes/no
p is false and q is false?	yes/no

- examples:

$$\begin{aligned} \beta(\neg p) &= 0011 &&= \langle \text{ no, no, yes, yes } \rangle \\ \beta(p \leftrightarrow q) &= 1001 &&= \langle \text{ yes, no, no, yes } \rangle \\ \beta(p \rightarrow q) &= 1011 &&= \langle \text{ yes, no, yes, yes } \rangle \end{aligned}$$

from $2^3 = 8$ bitstrings of length 3 to $2^4 = 16$ bitstrings of length 4

Modal Logic S5	Propositional Logic	bitstrings level 1	bitstrings level 3	Propositional Logic	Modal Logic S5
$\Box p$	$p \wedge q$	1000	0111	$\neg(p \wedge q)$	$\neg\Box p$
$\neg\Box p \wedge p$	$\neg(p \rightarrow q)$	0100	1011	$p \rightarrow q$	$\Box p \vee \neg p$
$\Diamond p \wedge \neg p$	$\neg(p \leftarrow q)$	0010	1101	$p \leftarrow q$	$\neg\Diamond p \vee p$
$\neg\Diamond p$	$\neg(p \vee q)$	0001	1110	$p \vee q$	$\Diamond p$

Modal Logic S5	Propositional Logic	bitstrings level 2/0	bitstrings level 2/4	Propositional Logic	Modal Logic S5
p	p	1100	0011	$\neg p$	$\neg p$
$\Box p \vee (\Diamond p \wedge \neg p)$	q	1010	0101	$\neg q$	$\neg\Diamond p \vee (\neg\Box p \wedge p)$
$\Box p \vee \neg\Diamond p$	$p \leftrightarrow q$	1001	0110	$\neg(p \leftrightarrow q)$	$\neg\Box p \wedge \Diamond p$
$\Box p \wedge \neg\Box p$	$p \wedge \neg p$	0000	1111	$p \vee \neg p$	$\Box p \vee \neg\Box p$

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- recall: given a logic S, two **formulas** φ, ψ are

S-*contradictory* (CD_S)	iff	$\models_S \neg(\varphi \wedge \psi)$	and	$\models_S \neg(\neg\varphi \wedge \neg\psi)$
S-*contrary* (C_S)	iff	$\models_S \neg(\varphi \wedge \psi)$	and	$\not\models_S \neg(\neg\varphi \wedge \neg\psi)$
S-*subcontrary* (SC_S)	iff	$\not\models_S \neg(\varphi \wedge \psi)$	and	$\models_S \neg(\neg\varphi \wedge \neg\psi)$
in S-*subalternation* (SA_S)	iff	$\models_S \varphi \to \psi$	and	$\not\models_S \psi \to \varphi$

- $\{0,1\}^n$ is a Boolean algebra, so it can be used to characterise the Aristotelian relations: two **bitstrings** b_1, b_2 of length n are

n-*contradictory* (CD_n)	iff	$b_1 \wedge b_2 = 0 \cdots 0$	and	$b_1 \vee b_2 = 1 \cdots 1$
n-*contrary* (C_n)	iff	$b_1 \wedge b_2 = 0 \cdots 0$	and	$b_1 \vee b_2 \neq 1 \cdots 1$
n-*subcontrary* (SC_n)	iff	$b_1 \wedge b_2 \neq 0 \cdots 0$	and	$b_1 \vee b_2 = 1 \cdots 1$
in n-*subalternation* (SA_n)	iff	$b_1 \wedge b_2 = b_1$	and	$b_1 \vee b_2 \neq b_1$

- φ and ψ stand in some Aristotelian relation (defined for S) iff $\beta(\varphi)$ and $\beta(\psi)$ stand in that same relation (defined for bitstrings)
- β maps formulas from S to bitstrings, preserving Aristotelian structure

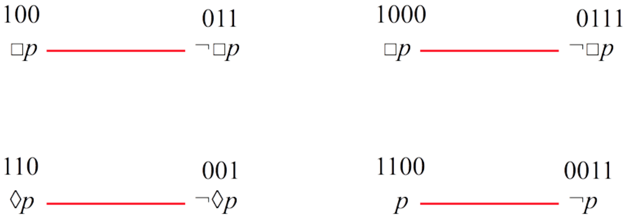
Aristotelian diagrams: algebraic characterisation

- let $\mathbb{B} = \langle B, \wedge, \vee, \neg, \top, \bot \rangle$ be an arbitrary Boolean algebra
- consider a non-empty fragment $\mathcal{F} \subseteq B$ such that
  - $\top, \bot \notin \mathcal{F}$
  - $\mathcal{F}$ is **closed under $\mathbb{B}$-complementation**: if $x \in \mathcal{F}$ then $\neg x \in \mathcal{F}$

- an **Aristotelian diagram for $\mathcal{F}$ in $\mathbb{B}$** is a diagram that visualizes an edge-labeled graph $\mathcal{G}$
  - the vertices of $\mathcal{G}$ are the elements of $\mathcal{F}$
  - the edges of $\mathcal{G}$ are labeled by the relations of $\mathcal{AG}_{\mathbb{B}}$ between those elements
  - if $x, y \in \mathcal{F}$ stand in some Aristotelian relation in $\mathbb{B}$, then this is visualized according to the code

- let S be an appropriate logical system (Boolean + $\models$)
- consider a non-empty fragment $\mathcal{F} \subseteq \mathcal{L}_\mathsf{S}$ such that
  - every formula $\varphi \in \mathcal{F}$ is S-**contingent**: $\not\models_\mathsf{S} \varphi$ and $\not\models_\mathsf{S} \neg\varphi$
  - $\mathcal{F}$ is **closed under negation** (up to $\equiv_\mathsf{S}$):
    if $\varphi \in \mathcal{F}$ then $\exists \psi \in \mathcal{F} : \psi \equiv_\mathsf{S} \neg\varphi$
  - the formulas in $\mathcal{F}$ are **pairwise non**-S-**equivalent**:
    if $\varphi, \psi \in \mathcal{F}$ are distinct, then $\varphi \not\equiv_\mathsf{S} \psi$

- an **Aristotelian diagram for** $\mathcal{F}$ **in** S is a diagram that visualizes an edge-labeled graph $\mathcal{G}$
  - the vertices of $\mathcal{G}$ are the elements of $\mathcal{F}$
  - the edges of $\mathcal{G}$ are labeled by the relations of $\mathcal{AG}_\mathsf{S}$ between those elements
  - if $\varphi, \psi \in \mathcal{F}$ stand in some Aristotelian relation in S, then this is visualized according to the code



- **PCD = pair of contradictories**
- a PCD is the **smallest possible** Aristotelian diagram
  - no Aristotelian diagrams with a single formula
  - because of the requirement that they be closed under negation
- PCDs are the **building blocks** for all larger Aristotelian diagrams

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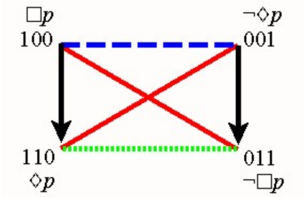
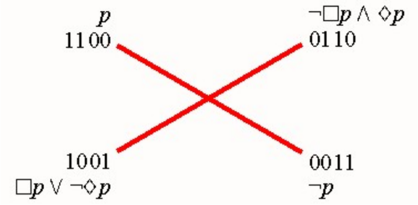




classical square
square of opposition

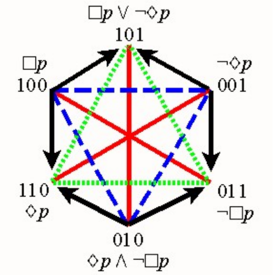
degenerate square
unconnectedness square
X of opposition

2 PCDs

2 subalternations (SA)
1 contrariety (C)
1 subcontrariety (SC)

2 PCDs

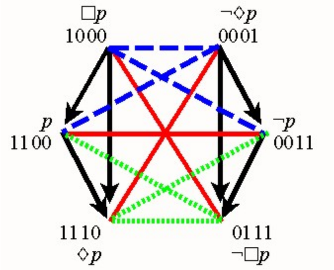
4 × unconnectedness (U)



Jacoby-Sesmat-Blanché hexagon
JSB hexagon

3 PCDs

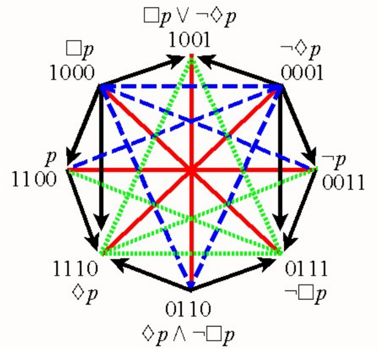
6 subalternations (SA)
3 contrarieties (C)
3 subcontrarieties (SC)



Sherwood-Czezowski hexagon
SC hexagon

3 PCDs

6 subalternations (SA)
3 contrarieties (C)
3 subcontrarieties (SC)

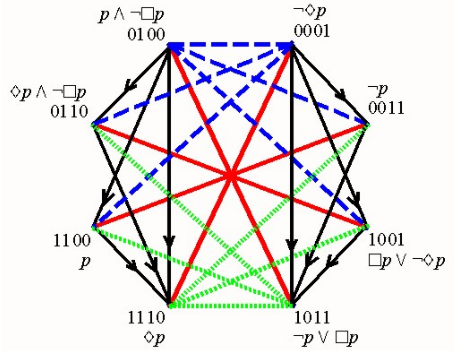


Béziau octagon

4 PCDs

10 SAs & 5 Cs & 5 SCs

4 × unconnectedness (U)



Buridan octagon

4 PCDs

10 SAs & 5 Cs & 5 SCs

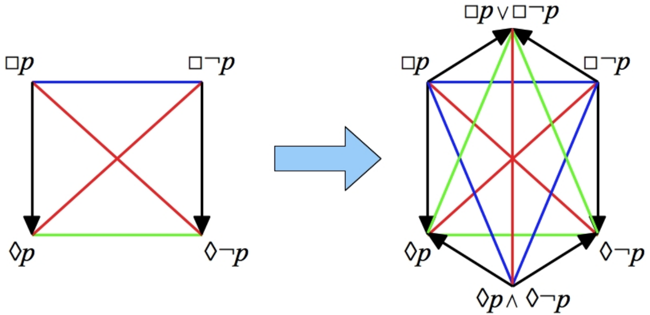
4 × unconnectedness (U)

- **Boolean closure of a fragment $\mathcal{F}$:**
  - the smallest Boolean algebra that contains $\mathcal{F}$
  - contains all Boolean combinations of formulas from $\mathcal{F}$
  - notation: $\mathbb{B}(\mathcal{F})$
  - contains 2^n formulas, for some natural number n

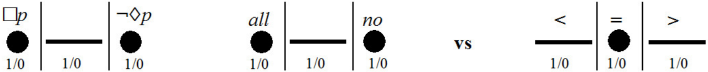
- **Boolean closure of an Aristotelian diagram for $\mathcal{F}$ in** S:
  - Aristotelian diagram for $\mathbb{B}(\mathcal{F})$ in S
  - note: Aristotelian diagram, so only S-contingent formulas
  - contains $2^n - 2$ formulas, for some natural number n

- some examples:
  - the Boolean closure of a classical square is a JSB hexagon
    $\Rightarrow 2^3 - 2 = 6$ contingent Boolean combinations
  - the Boolean closure of a degenerate square is a rhombic dodecahedron
  - the Boolean closure of an SC hexagon is a rhombic dodecahedron
    $\Rightarrow 2^4 - 2 = 14$ contingent Boolean combinations

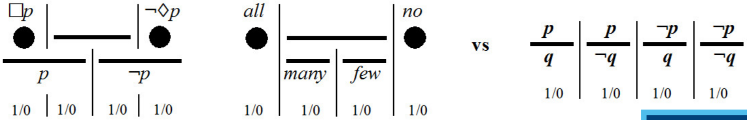
- the Boolean closure of a classical square is a JSB hexagon



- logical and diagrammatic effectiveness ☞ next lectures
- linguistic and cognitive effectiveness:
  bitstrings generate new questions about
  - the **linguistic/cognitive aspects** of the expressions they encode
  - the **relative weight/strength** of individual bit positions inside bitstrings
  - the underlying **scalar/linear structure** of the conceptual domain
- edges versus center in bitstrings of length 3



- bitstrings of length 4 as refinements/coarsenings of bitstrings of length 3



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- **no systematic method** for establishing a bitstring semantics for any fragment $\mathcal{F}$ in any logical system S
  ☞ final part of lecture 1
- no good grasp of the intricate **interplay between Aristotelian and Boolean structure**
  ☞ first part of lecture 4
- no good grasp of the **logic-sensitivity of the Aristotelian relations**
  ☞ second part of lecture 4
- to overcome these limitations: develop more mathematically precise approach to bitstring semantics

- $\{0,1\}^n$ forms a Boolean algebra (bitstrings of length n)
  - $\wedge$, $\vee$ and $\neg$ are defined componentwise
  - top element: $1 \cdots 1$
  - bottom element: $0 \cdots 0$

- we can define the Aristotelian relations between bitstrings: two bitstrings $b_1, b_2 \in \{0,1\}^n$ are

n-contradictory (CD_n)	iff	$b_1 \wedge b_2 = 0 \cdots 0$	and	$b_1 \vee b_2 = 1 \cdots 1$
n-contrary (C_n)	iff	$b_1 \wedge b_2 = 0 \cdots 0$	and	$b_1 \vee b_2 \neq 1 \cdots 1$
n-subcontrary (SC_n)	iff	$b_1 \wedge b_2 \neq 0 \cdots 0$	and	$b_1 \vee b_2 = 1 \cdots 1$
in n-subalternation (SA_n)	iff	$b_1 \wedge b_2 = b_1$	and	$b_1 \vee b_2 \neq b_1$

- the Aristotelian geometry for bitstrings of length n:

$$\mathcal{AG}_n := \{CD_n, C_n, SC_n, SA_n\}$$

- the setup:
  - logical systems $\mathsf{S}_1, \mathsf{S}_2$ and natural numbers n_1, n_2
  - $x \in \{\mathsf{S}_1, n_1\}$ and $y \in \{\mathsf{S}_2, n_2\}$
  - $\mathcal{F}_x$ is a finite set of formulas of system x/bitstrings of length x
  - $\mathcal{F}_y$ is a finite set of formulas of system y/bitstrings of length y

- we will define functions from $\mathcal{F}_x$ to $\mathcal{F}_y$

- this encompasses **four cases**:
  - from formulas of S_1 to formulas of S_2
  - from formulas of S_1 to bitstrings of length n_2
  - from bitstrings of length n_1 to formulas of S_2
  - from bitstrings of length n_1 to bitstrings of length n_2

- the setup:
  - logical systems $\mathsf{S}_1, \mathsf{S}_2$ and natural numbers n_1, n_2
  - $x \in \{\mathsf{S}_1, n_1\}$ and $y \in \{\mathsf{S}_2, n_2\}$
  - $\mathcal{F}_x$ is a finite set of formulas of system x/bitstrings of length x
  - $\mathcal{F}_y$ is a finite set of formulas of system y/bitstrings of length y

- a bijection $\gamma\colon \mathcal{F}_x \to \mathcal{F}_y$ is an **Aristotelian isomorphism** iff for all Aristotelian relations $R_x \in \mathcal{AG}_x$ and corresponding $R_y \in \mathcal{AG}_y$, and for all $\varphi, \psi \in \mathcal{F}_x$, it holds that $R_x(\varphi, \psi)$ iff $R_y(\gamma(\varphi), \gamma(\psi))$

- a bijection $\gamma\colon \mathcal{F}_x \to \mathcal{F}_y$ is a **Boolean isomorphism** iff there exists some Boolean algebra isomorphism $f\colon \mathbb{B}(\mathcal{F}_x) \to \mathbb{B}(\mathcal{F}_y)$ such that $\gamma = f \restriction \mathcal{F}_x$

  (recall that $\mathbb{B}(\mathcal{F})$ is the Boolean closure of $\mathcal{F}$)

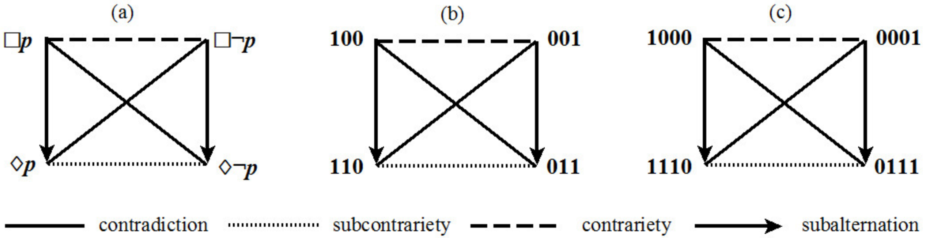
- since the Aristotelian relations are defined in purely Boolean terms, the Aristotelian structure of a fragment is entirely determined by its Boolean structure

- **lemma**: for any $\gamma\colon \mathcal{F}_x \to \mathcal{F}_y$:
  if γ is a Boolean isomorphism, then γ is an Aristotelian isomorphism

- a **bitstring semantics** for $\mathcal{F}_x$ is a Boolean algebra isomorphism $\beta\colon \mathbb{B} \to \{0,1\}^n$, where $\mathbb{B}$ is some Boolean algebra that contains $\mathcal{F}_x$ (not necessarily the smallest one)

- **lemma**: every bitstring semantics $\beta\colon \mathbb{B} \to \{0,1\}^n$ is an Aristotelian isomorphism

- fragment $\mathcal{F}$ of S5-formulas: $\{\Box p, \Diamond p, \Box\neg p, \Diamond\neg p\}$
- two Boolean algebras that contain $\mathcal{F}$:
  - $\mathbb{B}_3$, which has atoms $\Box p, \Diamond p \wedge \Diamond\neg p$ and $\Box\neg p$ (note: $\mathbb{B}_3 = \mathbb{B}(\mathcal{F})$)
  - $\mathbb{B}_4$, which has atoms $\Box p, p \wedge \Diamond\neg p, \neg p \wedge \Diamond p$ and $\Box\neg p$
- two bitstring semantics for $\mathcal{F}$:
  - $\beta_3 : \mathbb{B}_3 \to \{0,1\}^3$
  - $\beta_4 : \mathbb{B}_4 \to \{0,1\}^4$



- let S be a logical system with Boolean operators and a semantics $\models$, and consider $\mathcal{F} = \{\varphi_1, \ldots, \varphi_m\} \subseteq \mathcal{L}_\mathsf{S}$

- the **partition of** S **induced by** $\mathcal{F}$ is

  $\Pi_\mathsf{S}(\mathcal{F}) := \{\alpha \in \mathcal{L}_\mathsf{S} \mid \alpha \equiv_\mathsf{S} \pm\varphi_1 \wedge \cdots \wedge \pm\varphi_m$, and α is S-consistent$\}$

- $\pm\varphi$ stands for either φ or $\neg\varphi$; α should be read up to $\equiv_\mathsf{S}$

- the formulas $\alpha \in \Pi_\mathsf{S}(\mathcal{F})$ are called **anchor formulas**
  - in principle, equivalent to a conjunction of $m = |\mathcal{F}|$ conjuncts
  - can often be simplified, e.g. when $\neg\varphi_i \equiv_\mathsf{S} \varphi_j$ for some $\varphi_i, \varphi_j \in \mathcal{F}$

- $\Pi_\mathsf{S}(\mathcal{F})$ is a **partition** of (the class of all models of) S:
  - $\models_\mathsf{S} \neg(\alpha_i \wedge \alpha_j)$ for distinct $\alpha_i, \alpha_j \in \Pi_\mathsf{S}(\mathcal{F})$ (mutually exclusive)
  - $\models_\mathsf{S} \bigvee \Pi_\mathsf{S}(\mathcal{F})$ (jointly exhaustive)

- first-order logic (FOL), fragment $\mathcal{F} := \{\forall x Px,\ \exists x Px,\ \neg Pa\}$
- let's compute $\Pi_{\mathsf{FOL}}(\mathcal{F})$, the partition of FOL induced by $\mathcal{F}$
- there are $2^{|\mathcal{F}|} = 2^3 = 8$ relevant conjunctions

1.	$\forall x Px$	$\wedge$	$\exists x Px$	$\wedge$	$\neg Pa$	$\rightsquigarrow$	FOL-inconsistent
2.	$\forall x Px$	$\wedge$	$\exists x Px$	$\wedge$	$\neg\neg Pa$	$\rightsquigarrow$	$\forall x Px$
3.	$\forall x Px$	$\wedge$	$\neg\exists x Px$	$\wedge$	$\neg Pa$	$\rightsquigarrow$	FOL-inconsistent
4.	$\forall x Px$	$\wedge$	$\neg\exists x Px$	$\wedge$	$\neg\neg Pa$	$\rightsquigarrow$	FOL-inconsistent
5.	$\neg\forall x Px$	$\wedge$	$\exists x Px$	$\wedge$	$\neg Pa$	$\rightsquigarrow$	$\neg Pa \wedge \exists x Px$
6.	$\neg\forall x Px$	$\wedge$	$\exists x Px$	$\wedge$	$\neg\neg Pa$	$\rightsquigarrow$	$Pa \wedge \neg\forall x Px$
7.	$\neg\forall x Px$	$\wedge$	$\neg\exists x Px$	$\wedge$	$\neg Pa$	$\rightsquigarrow$	$\neg\exists x Px$
8.	$\neg\forall x Px$	$\wedge$	$\neg\exists x Px$	$\wedge$	$\neg\neg Pa$	$\rightsquigarrow$	FOL-inconsistent

- $\Pi_{\mathsf{FOL}}(\mathcal{F}) = \{\forall x Px,\ \neg Pa \wedge \exists x Px,\ Pa \wedge \neg\forall x Px,\ \neg\exists x Px\}$

- given partitions Π_1 and Π_2:
  - Π_1 is a **refinement** of Π_2 iff
    for all $\alpha \in \Pi_1$ there exists $\alpha' \in \Pi_2$ such that $\models_{\mathsf{S}} \alpha \to \alpha'$
  - the **meet** of Π_1 and Π_2 is defined as follows:
    $\Pi_1 \wedge_{\mathsf{S}} \Pi_2 := \{\gamma_1 \wedge \gamma_2 \mid \gamma_1 \in \Pi_1, \gamma_2 \in \Pi_2, \text{ and } \gamma_1 \wedge \gamma_2 \text{ is S-consistent}\}$
  - note: $\Pi_1 \wedge_{\mathsf{S}} \Pi_2$ is the coarsest common refinement of Π_1 and Π_2

- **lemma**: if $\mathcal{F}_1 \cup \mathcal{F}_2 = \mathcal{F}$, then $\Pi_{\mathsf{S}}(\mathcal{F}_1) \wedge_{\mathsf{S}} \Pi_{\mathsf{S}}(\mathcal{F}_2) = \Pi_{\mathsf{S}}(\mathcal{F})$

- **lemma**: if $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then $\Pi_{\mathsf{S}}(\mathcal{F}_2)$ is a refinement of $\Pi_{\mathsf{S}}(\mathcal{F}_1)$

- given two logics S_1 and S_2 (with the same language $\mathcal{L}$), we say that S_2 is **stronger** than S_1 iff for all $\varphi \in \mathcal{L}$: if $\models_{\mathsf{S}_1} \varphi$ then $\models_{\mathsf{S}_2} \varphi$

- **lemma**: if S_2 is stronger than S_1, then
  $\Pi_{\mathsf{S}_2}(\mathcal{F}) = \{\alpha \in \Pi_{\mathsf{S}_1}(\mathcal{F}) \mid \alpha \text{ is } \mathsf{S}_2\text{-consistent}\}$

- logic S, fragment $\mathcal{F}$ and partition $\Pi_{\mathsf{S}}(\mathcal{F}) = \{\alpha_1, \ldots, \alpha_n\}$
- **lemma**: for all $\varphi \in \mathbb{B}(\mathcal{F})$:
  - for all $\alpha_i \in \Pi_{\mathsf{S}}(\mathcal{F})$ we have $\models_{\mathsf{S}} \alpha_i \to \varphi$ or $\models_{\mathsf{S}} \alpha_i \to \neg\varphi$, but not both
  - $\varphi \equiv_{\mathsf{S}} \bigvee\{\alpha \in \Pi_{\mathsf{S}}(\mathcal{F}) \mid \models_{\mathsf{S}} \alpha \to \varphi\}$
- for every $\varphi \in \mathbb{B}(\mathcal{F})$, we define a bitstring $\beta_{\mathsf{S}}^{\mathcal{F}}(\varphi) \in \{0,1\}^n$ as follows:

  for each bit position $1 \leq i \leq n$: $[\beta_{\mathsf{S}}^{\mathcal{F}}(\varphi)]_i := \begin{cases} 1 & \text{if } \models_{\mathsf{S}} \alpha_i \to \varphi, \\ 0 & \text{if } \models_{\mathsf{S}} \alpha_i \to \neg\varphi. \end{cases}$
- **lemma**: for all $\varphi \in \mathbb{B}(\mathcal{F})$ we have $\varphi \equiv_{\mathsf{S}} \bigvee\{\alpha_i \in \Pi_{\mathsf{S}}(\mathcal{F}) \mid [\beta_{\mathsf{S}}^{\mathcal{F}}(\varphi)]_i = 1\}$
- relativized disjunctive normal form: φ is rewritten as
  - a disjunction of anchor formulas, which are themselves
  - conjunctions of (possibly negated) formulas $\pm\varphi_j \in \mathcal{F}$

- for every $\varphi \in \mathbb{B}(\mathcal{F})$, we have bitstring $\beta_{\mathsf{S}}^{\mathcal{F}}(\varphi) \in \{0,1\}^n = \{0,1\}^{|\Pi_{\mathsf{S}}(\mathcal{F})|}$
- turn this into a **function** $\beta_{\mathsf{S}}^{\mathcal{F}} : \mathbb{B}(\mathcal{F}) \to \{0,1\}^{|\Pi_{\mathsf{S}}(\mathcal{F})|}$

- **theorem**: $\beta_{\mathsf{S}}^{\mathcal{F}}$ is a bitstring semantics for $\mathcal{F}$

- **corollary**: $|\mathbb{B}(\mathcal{F})| = 2^{|\Pi_{\mathsf{S}}(\mathcal{F})|}$

- **corollary**: $\beta_{\mathsf{S}}^{\mathcal{F}}$ is an Aristotelian isomorphism

- **corollary**: $\beta_{\mathsf{S}}^{\mathcal{F}}$ is a minimal bitstring semantics for $\mathcal{F}$:

  every other bitstring semantics for $\mathcal{F}$ is
  either a permutation variant of $\beta_{\mathsf{S}}^{\mathcal{F}}$
  or makes use of bitstrings of length $> |\Pi_{\mathsf{S}}(\mathcal{F})|$

- fragment size $m := |\mathcal{F}|$ and bitstring length $n := |\Pi_{\mathsf{S}}(\mathcal{F})|$
- **theorem**:

  (1) we can bound m in terms of n: $\lceil \log_2(n) \rceil \leq m \leq 2^n$

  (2) we can bound n in terms of m: $\lceil \log_2(m) \rceil \leq n \leq 2^m$

- (1) and (2) can be seen as each other's inverses
- all these bounds are tight
- **theorem** (special case, but very relevant for logical geometry):
  if $\mathcal{F}$ only contains S-contingent formulas and is closed under negation:

  (1') bound m in terms of n: $2\lceil \log_2(n) \rceil \leq m \leq 2^n - 2$

  (2') bound n in terms of m: $\lceil \log_2(m+2) \rceil \leq n \leq 2^{\frac{m}{2}}$

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Thank you!

Questions?

More info: `www.logicalgeometry.org`