Introduction to Logical Geometry

1. Basic Concepts and Bitstring Semantics

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Practicalities

- Lecturers:
  - Lorenz Demey
    - Primary background in logic and philosophy
    - Center for Logic and Philosophy of Science, KU Leuven
    - [http://www.lorenzdemey.eu](http://www.lorenzdemey.eu)
  - Hans Smessaert
    - Primary background in linguistics
    - Research Group Formal and Computational Linguistics, KU Leuven

- Course website:
  - Course slides
  - Background readings
Who are you?

- what’s your academic background?
  - philosophy
  - logic
  - linguistics
  - mathematics
  - computer science

Introduction to Logical Geometry – Part 1
Motivating examples

- logical geometry ~ the systematic study of Aristotelian diagrams

- what are Aristotelian diagrams?
  - later: precise definition
  - now: some motivating examples

- some general trends to pay attention to:
  - long history, but still used today
  - applications in logic and philosophy, but also in many other disciplines
  - not just for teaching purposes, but also in research contexts
Square of opposition

- oldest and most well-known example of an Aristotelian diagram
- the square of opposition for the categorical statements from syllogistics
  - relations: Aristotle (4th century BCE)
  - diagram: Apuleius of Madaura (2nd century CE), Boethius (5th century CE)
• epic poem: *Der Wälsche Gast*
• visual representation of the seven liberal arts
Peter Abelard (1079 – 1142)

- square for the quantifiers from the categorical statements (all, some, no)
- also a square for the dual quantifiers (both, either, neither)
Peter of Spain (13th century)

- squares for the quantifiers and the modalities
- within each vertex: duality behavior
  - every man runs
  - no man does not run
  - not some man does not run
modal syllogistics: propositions with quantifiers and modalities
‘figura completa’, but also ‘figura incompleta’
John Buridan (1300 – 1358)

- integrates several squares into one ‘magna figura’
- for modal syllogistics, but also for other types of propositions
Nicole Oresme (1323 – 1382)

- (proto-scientific) cosmology: *Livre du Ciel et du Monde*
- an ‘extended’ square: add the conjunction of the two lower corners

```
always possible to be
always possible not to be
not always possible not to be
not always possible to be
the intermediate
```
Jacques Lefèvre d’Étaples (1455 – 1536)

- analogy:
  - a square for propositions // a square for properties
  - ‘cannot be true together’ // ‘cannot be instantiated together’
- squares for quantifiers, propositional connectives, modalities, temporal and spatial adverbs
“In the nineteenth century, the apparently most widely used textbook in Britain and America” (Parsons, 2017)

usual square for the categorical statements

three types of matter (connection between subject and predicate): [n]ecessary, [i]mpossible and [c]ontingent
a square of opposition in Begriffschrift notation

note the mistake: ‘conträr’ \(\sim\) ‘subconträr’
octagon for the categorical statements with subject negation
20th-century/contemporary philosophers and logicians

- Ruth Barcan Marcus
- Arthur Prior
- Hans Reichenbach
- Richard Hare
- H. L. A. Hart (cf. figure)
- Roderick Chisholm
- Ernest Sosa
Applications beyond logic and philosophy

- linguistics
  - semantics (generalized quantifiers) (Dag Westerståhl)
  - pragmatics (implicatures) (Laurence Horn)
  - typology (lexicalization) (Debra Ziegeler)

On the empty O-corner of the Aristotelian Square: A view from Singapore English

Debra Ziegeler*

Université Sorbonne Nouvelle Paris 3, France
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Available online 18 May 2017

Introduction to Logical Geometry – Part 1
Applications beyond logic and philosophy

- cognitive science
  - psychology of reasoning
  - emotions research
  - neuroscience

(Stephen Newstead, Richard Griggs)
(Olivier Massin)
(Camillo Porcaro et al.)

Drawing Inferences from Quantified Statements: A Study of the Square of Opposition

Universal vs. particular reasoning: a study with neuroimaging techniques

Introduction to Logical Geometry – Part 1
Applications beyond logic and philosophy

- computer science (knowledge representation)
  - formal concept analysis
  - rough set theory
  - formal argumentation theory

The Cube of Opposition -
A Structure underlying many Knowledge Representation Formalisms
Aristotelian diagrams have been used for a very long time (including today) in a wide variety of disciplines (not just logic and philosophy).

Aristotelian diagrams constitute a language for a broad (transdisciplinary and transhistorical) community of researchers who deal with logical reasoning.

Logical geometry ~ the linguistics that systematically studies the language of Aristotelian diagrams.

Two fundamental aspects of any language:

- **Syntax**: form, representation
- **Semantics**: meaning, what is represented

~→ ‘geometry’

~→ ‘logical’
perspective shift:

- in a typical application:
  Aristotelian diagrams are used (= tool)
  to analyze some linguistic, logical, conceptual phenomenon (= object)

- in logical geometry:
  Aristotelian diagrams are themselves the primary objects of study,
  analyzed using a variety of tools (bitstring analysis, group theory, etc.)

this has led to an elaborate (and growing) elegant theory
(regardless of the multitude of applications)

double motivation for logical geometry:

- Aristotelian diagrams as objects of independent interest
- Aristotelian diagrams as a widely-used language
other types of logic diagrams:

- Hasse diagrams
- Euler/Venn diagrams
- duality diagrams

since the 1990s: diagrammatic reasoning

two courses at ESSLLI 2017:

- Caught in the Spiders’ Diagrammatic Reasoning Web – The Euler/Spider Diagram Family of Formal Reasoning Systems
- Picturing Quantum Processes

We provide a self-contained introduction to quantum theory . . . This course is unique in our use of a diagrammatic language throughout. Far from simple visual aids, the diagrams we use are mathematical objects in their own right
1. Basic Concepts and Bitstring Semantics

   - Aristotelian, Opposition, Implication and Duality Relations

   - Informational Equivalence, Symmetry and Distance

4. Abstract-Logical Properties of Aristotelian Diagrams, Part II
   - Boolean Structure and Logic-Sensitivity

5. Case Studies and Philosophical Outlook
1. Basic Concepts and Bitstring Semantics

   - Aristotelian, Opposition, Implication and Duality Relations

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4. Abstract-Logical Properties of Aristotelian Diagrams, Part II
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5. Case Studies and Philosophical Outlook
two propositions are said to be

contradictory (CD) \( \iff \) they cannot be true together and they cannot be false together

contrary (C) \( \iff \) they cannot be true together but they can be false together

subcontrary (SC) \( \iff \) they can be true together but they cannot be false together

in subalternation (SA) \( \iff \) the first proposition entails the second but the second doesn’t entail the first
let $S$ be a logical system with
- the usual Boolean connectives ($\land, \lor, \neg, \rightarrow$)
- a model-theoretic semantics ($\models$)

two formulas $\varphi, \psi \in \mathcal{L}_S$ are said to be
- **S-contradictory** ($CD_S$) iff $\models_S \neg(\varphi \land \psi)$ and $\models_S \neg(\neg\varphi \land \neg\psi)$
- **S-contrary** ($C_S$) iff $\models_S \neg(\varphi \land \psi)$ and $\not\models_S \neg(\neg\varphi \land \neg\psi)$
- **S-subcontrary** ($SC_S$) iff $\not\models_S \neg(\varphi \land \psi)$ and $\models_S \neg(\neg\varphi \land \neg\psi)$
- in **S-subalternation** ($SA_S$) iff $\models_S \varphi \rightarrow \psi$ and $\not\models_S \psi \rightarrow \varphi$

the Aristotelian geometry for $S$: $\mathcal{AG}_S := \{CD_S, C_S, SC_S, SA_S\}$

the Aristotelian relations are defined up to logical equivalence:
- suppose that $\varphi \equiv_S \varphi'$ and $\psi \equiv_S \psi'$
- then for all $R \in \mathcal{AG}_S$: $R_S(\varphi, \psi) \Leftrightarrow R_S(\varphi', \psi')$
Aristotelian relations: algebraic characterisation

- let $\mathbb{B} = \langle B, \land, \lor, \neg, \top, \bot \rangle$ be an arbitrary Boolean algebra

- two elements $x, y \in B$ are said to be
  
  $\mathbb{B}$-contradictory ($CD_B$) iff $x \land y = \bot$ and $x \lor y = \top$
  
  $\mathbb{B}$-contrary ($C_B$) iff $x \land y = \bot$ and $x \lor y \neq \top$
  
  $\mathbb{B}$-subcontrary ($SC_B$) iff $x \land y \neq \bot$ and $x \lor y = \top$
  
  in $\mathbb{B}$-subalternation ($SA_B$) iff $\neg x \lor y = \top$ and $x \lor \neg y \neq \top$

- the Aristotelian geometry for $\mathbb{B}$: $AG_B := \{ CD_B, C_B, SC_B, SA_B \}$

- thanks to this abstract characterisation, Aristotelian relations can be defined between formulas/statements and between sets/concepts
  
  - cf. Lefèvre d’Étaples’s ‘analogia’ between two squares of oppositions
  
  - Keynes, 1906: “These seven possible relations between propositions (taken in pairs) will be found to be precisely analogous to the seven possible relations between classes (taken in pairs)”
first concrete instance of the algebraic characterisation: Aristotelian relations in a Lindenbaum-Tarski algebra

S-equivalence classes of formulas: \([\varphi]_S := \{\psi \in \mathcal{L}_S \mid \varphi \equiv_S \psi\}\)

let \(\mathbb{B}(S)\) be the Lindenbaum-Tarski algebra of the logical system \(S\)

two equivalence classes \([\varphi]_S, [\psi]_S\) are said to be

\(\mathbb{B}(S)\)-contradictory iff \([\varphi]_S \land [\psi]_S = \bot\) and \([\varphi]_S \lor [\psi]_S = \top\)

\(\mathbb{B}(S)\)-contrary iff \([\varphi]_S \land [\psi]_S = \bot\) and \([\varphi]_S \lor [\psi]_S \neq \top\)

\(\mathbb{B}(S)\)-subcontrary iff \([\varphi]_S \land [\psi]_S \neq \bot\) and \([\varphi]_S \lor [\psi]_S = \top\)

in \(\mathbb{B}(S)\)-subalternation iff \([\neg \varphi]_S \lor [\psi]_S = \top\) and \([\varphi]_S \lor [\neg \psi]_S \neq \top\)

this characterisation essentially corresponds to the model-theoretic one: e.g. \(\varphi\) and \(\psi\) are \(S\)-contrary iff \([\varphi]_S\) and \([\psi]_S\) are \(\mathbb{B}(S)\)-contrary
second concrete instance of the algebraic characterisation: Aristotelian relations in a Boolean algebra of sets

let $\mathcal{B} = \langle B, \cap, \cup, \setminus, D, \emptyset \rangle$ be a Boolean algebra of sets

two sets $X, Y \in B$ are said to be

$\mathcal{B}$-contradictory iff $X \cap Y = \emptyset$ and $X \cup Y = D$

$\mathcal{B}$-contrary iff $X \cap Y = \emptyset$ and $X \cup Y \neq D$

$\mathcal{B}$-subcontrary iff $X \cap Y \neq \emptyset$ and $X \cup Y = D$

in $\mathcal{B}$-subalternation iff $(D \setminus X) \cup Y = D$ and $X \cup (D \setminus Y) \neq D$
informal characterisation:

two propositions \( \varphi, \psi \) are said to be unconnected iff

(i) \( \varphi \) and \( \psi \) can be true together and
(ii) \( \varphi \) does not entail \( \psi \) and
(iii) \( \psi \) does not entail \( \varphi \) and
(iv) \( \varphi \) and \( \psi \) can be false together

together, these four conditions imply that \( \varphi \) and \( \psi \) do not stand in any Aristotelian relation:

- condition (i) implies that \( \varphi \) and \( \psi \) are neither CD nor C
- condition (ii) implies that there is no SA from \( \varphi \) to \( \psi \)
- condition (iii) implies that there is no SA from \( \psi \) to \( \varphi \)
- condition (iv) implies that \( \varphi \) and \( \psi \) are neither CD nor SC
Aristotelian relations and unconnectedness

- **model-theoretic characterisation:**
  two formulas \( \varphi, \psi \) are said to be \( \text{S-unconnected} \) iff

  (i) \( \not\models_S \neg(\varphi \land \psi) \) and
  (ii) \( \not\models_S \varphi \rightarrow \psi \) and
  (iii) \( \not\models_S \psi \rightarrow \varphi \) and
  (iv) \( \not\models_S \neg(\neg\varphi \land \neg\psi) \)

- **algebraic characterisation:**
  two elements \( x, y \in B \) are said to be \( \text{B-unconnected} \) iff

  (i) \( x \land y \neq \bot \) and
  (ii) \( x \land \neg y \neq \bot \) and
  (iii) \( \neg x \land y \neq \bot \) and
  (iv) \( \neg x \land \neg y \neq \bot \)
first concrete instance: Lindenbaum-Tarski algebra:
two equivalence classes $[\varphi]_S$, $[\psi]_S$ are said to be $B(S)$-unconnected iff

(i) $[\varphi]_S \land [\psi]_S \neq \bot$ and
(ii) $[\varphi]_S \land [\neg \psi]_S \neq \bot$ and
(iii) $[\neg \varphi]_S \land [\psi]_S \neq \bot$ and
(iv) $[\neg \varphi]_S \land [\neg \psi]_S \neq \bot$

second concrete instance: Boolean algebra of sets:
two sets $X, Y \in B$ are said to be $B$-unconnected iff

(i) $X \cap Y \neq \emptyset$ and
(ii) $X \cap (D\setminus Y) \neq \emptyset$ and
(iii) $(D\setminus X) \cap Y \neq \emptyset$ and
(iv) $(D\setminus X) \cap (D\setminus Y) \neq \emptyset$
• bitstrings are finite **sequences of bits** (0/1), e.g. 10101011

• bitstrings can encode the denotations of formulas or expressions from:
  • **logical systems**: e.g. classical propositional logic, first-order logic, modal logic and public announcement logic
  • **lexical fields**: e.g. comparative quantification, subjective quantification, color terms and set inclusion relations

• each bit provides an **answer** to a meaningful (binary) **question**
  (origin: analysis of generalized quantifiers as sets of sets)

• note:
  • we use bitstrings to encode **formulas**, not **relations** between formulas
  • if a formula \( \varphi \) is encoded by the bitstring \( b \), we write \( \beta(\varphi) = b \)
  • \( [b]_i \) denotes the \( i^{th} \) bit position of the bitstring \( b \)
• each question concerns a component (point/interval) of a scalar structure that creates a partition of logical space

![Diagram showing all, no, and other states with corresponding 1/0 values]

• application to FOL/GQT: is \( Q(A, B) \) true if

<table>
<thead>
<tr>
<th>Question</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A \subseteq B ) ?</td>
<td>yes/no</td>
</tr>
<tr>
<td>( A \notin B ) and ( A \cap B \neq \emptyset ) ?</td>
<td>yes/no</td>
</tr>
<tr>
<td>( A \cap B = \emptyset ) ?</td>
<td>yes/no</td>
</tr>
</tbody>
</table>

\[ \beta(\text{all } A \text{ are } B) = 100 = \langle \text{yes, no, no} \rangle \]

• examples:

\[ \beta(\text{some but not all } A \text{ are } B) = 010 = \langle \text{no, yes, no} \rangle \]

\[ \beta(\text{not all } A \text{ are } B) = 011 = \langle \text{no, yes, yes} \rangle \]
application to the modal logic S5: is $\varphi$ true if

- $p$ is true in all possible worlds? yes/no
- $p$ is true in some but not in all possible worlds? yes/no
- $p$ is true in no possible worlds? yes/no

**examples:**

- $\beta(\Diamond p) = 110 = \langle \text{yes, yes, no} \rangle$
- $\beta(\Diamond p \land \Diamond \neg p) = 010 = \langle \text{no, yes, no} \rangle$
- $\beta(\Diamond \neg p) = 011 = \langle \text{no, yes, yes} \rangle$

<table>
<thead>
<tr>
<th>Modal Logic</th>
<th>GQT</th>
<th>level 1/0</th>
<th>level 2/3</th>
<th>GQT</th>
<th>Modal Logic</th>
</tr>
</thead>
<tbody>
<tr>
<td>necessary ($\Box p$)</td>
<td>all</td>
<td>100</td>
<td>011</td>
<td>not all</td>
<td>not necessary ($\neg \Box p$)</td>
</tr>
<tr>
<td>contingent ($\neg \Box p \land \Diamond p$)</td>
<td>some but not all</td>
<td>010</td>
<td>101</td>
<td>no or all</td>
<td>not contingent ($\Box p \lor \neg \Diamond p$)</td>
</tr>
<tr>
<td>impossible ($\neg \Diamond p$)</td>
<td>no</td>
<td>001</td>
<td>110</td>
<td>some</td>
<td>possible ($\Diamond p$)</td>
</tr>
<tr>
<td>contradiction ($\Box p \land \neg \Box p$)</td>
<td>some and no</td>
<td>000</td>
<td>111</td>
<td>some or no</td>
<td>tautology ($\Box p \lor \neg \Box p$)</td>
</tr>
</tbody>
</table>
**second** application to the modal logic S5: is $\varphi$ true if

- $p$ is true in all possible worlds? (yes/no)
- $p$ is true in the actual world but not in all possible worlds? (yes/no)
- $p$ is true in some possible worlds but not in the actual world? (yes/no)
- $p$ is true in no possible worlds? (yes/no)

**Examples:**

$$\beta(\Diamond p) = 1110 = \langle \text{yes, yes, yes, no} \rangle$$

$$\beta(\Diamond p \land \Diamond \neg p) = 0110 = \langle \text{no, yes, yes, no} \rangle$$

$$\beta(\Diamond \neg p) = 0111 = \langle \text{no, yes, yes, yes} \rangle$$
application to propositional logic: is $\varphi$ true if

- $p$ is true and $q$ is true? yes/no
- $p$ is true and $q$ is false? yes/no
- $p$ is false and $q$ is true? yes/no
- $p$ is false and $q$ is false? yes/no

\[
\beta(\neg p) = 0011 = \langle \text{no, no, yes, yes} \rangle
\]

\[
\beta(p \leftrightarrow q) = 1001 = \langle \text{yes, no, no, yes} \rangle
\]

\[
\beta(p \rightarrow q) = 1011 = \langle \text{yes, no, yes, yes} \rangle
\]
from $2^3 = 8$ bitstrings of length 3 to $2^4 = 16$ bitstrings of length 4

<table>
<thead>
<tr>
<th>Modal Logic S5</th>
<th>Propositional Logic</th>
<th>bitstrings level 1</th>
<th>bitstrings level 3</th>
<th>Propositional Logic</th>
<th>Modal Logic S5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\square p$</td>
<td>$p \land q$</td>
<td>1000</td>
<td>0111</td>
<td>$\neg(p \land q)$</td>
<td>$\neg \square p$</td>
</tr>
<tr>
<td>$\neg \square p \land p$</td>
<td>$\neg(p \to q)$</td>
<td>0100</td>
<td>1011</td>
<td>$p \to q$</td>
<td>$\square p \lor \neg p$</td>
</tr>
<tr>
<td>$\diamond p \land \neg p$</td>
<td>$\neg(p \to q)$</td>
<td>0010</td>
<td>1101</td>
<td>$p \to q$</td>
<td>$\neg \diamond p \lor p$</td>
</tr>
<tr>
<td>$\neg \diamond p$</td>
<td>$\neg(p \lor q)$</td>
<td>0001</td>
<td>1110</td>
<td>$p \lor q$</td>
<td>$\diamond p$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Modal Logic S5</th>
<th>Propositional Logic</th>
<th>bitstrings level 2/0</th>
<th>bitstrings level 2/4</th>
<th>Propositional Logic</th>
<th>Modal Logic S5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>$p$</td>
<td>1100</td>
<td>0011</td>
<td>$\neg p$</td>
<td>$\neg p$</td>
</tr>
<tr>
<td>$\square p \lor (\diamond p \land \neg p)$</td>
<td>$q$</td>
<td>1010</td>
<td>0101</td>
<td>$\neg q$</td>
<td>$\neg \diamond p \lor (\neg \square p \land p)$</td>
</tr>
<tr>
<td>$\square p \lor \neg \diamond p$</td>
<td>$p \to q$</td>
<td>1001</td>
<td>0110</td>
<td>$\neg(p \to q)$</td>
<td>$\neg \square p \land \diamond p$</td>
</tr>
<tr>
<td>$\square p \land \neg \square p$</td>
<td>$p \land \neg p$</td>
<td>0000</td>
<td>1111</td>
<td>$p \lor \neg p$</td>
<td>$\square p \lor \neg \square p$</td>
</tr>
</tbody>
</table>
recall: given a logic \( S \), two formulas \( \varphi, \psi \) are

- **S-contradictory** \((CD_S)\) iff \( \models_S \neg(\varphi \land \psi) \) and \( \models_S \neg(\neg\varphi \land \neg\psi) \)
- **S-contrary** \((C_S)\) iff \( \models_S \neg(\varphi \land \psi) \) and \( \not\models_S \neg(\neg\varphi \land \neg\psi) \)
- **S-subcontrary** \((SC_S)\) iff \( \not\models_S \neg(\varphi \land \psi) \) and \( \models_S \neg(\neg\varphi \land \neg\psi) \)
- **in S-subalternation** \((SA_S)\) iff \( \models_S \varphi \rightarrow \psi \) and \( \not\models_S \psi \rightarrow \varphi \)

\( \{0, 1\}^n \) is a Boolean algebra, so it can be used to characterise the Aristotelian relations: two **bitstrings** \( b_1, b_2 \) of length \( n \) are

- **n-contradictory** \((CD_n)\) iff \( b_1 \land b_2 = 0 \cdots 0 \) and \( b_1 \lor b_2 = 1 \cdots 1 \)
- **n-contrary** \((C_n)\) iff \( b_1 \land b_2 = 0 \cdots 0 \) and \( b_1 \lor b_2 \neq 1 \cdots 1 \)
- **n-subcontrary** \((SC_n)\) iff \( b_1 \land b_2 \neq 0 \cdots 0 \) and \( b_1 \lor b_2 = 1 \cdots 1 \)
- **in n-subalternation** \((SA_n)\) iff \( b_1 \land b_2 = b_1 \) and \( b_1 \lor b_2 \neq b_1 \)

\( \varphi \) and \( \psi \) stand in some Aristotelian relation (defined for \( S \)) iff

- \( \beta(\varphi) \) and \( \beta(\psi) \) stand in that same relation (defined for bitstrings)

- \( \beta \) maps formulas from \( S \) to bitstrings, preserving Aristotelian structure
let $\mathbb{B} = \langle B, \land, \lor, \neg, \top, \bot \rangle$ be an arbitrary Boolean algebra

consider a non-empty fragment $\mathcal{F} \subseteq B$ such that

- $\top, \bot \notin \mathcal{F}$
- $\mathcal{F}$ is closed under $\mathbb{B}$-complementation: if $x \in \mathcal{F}$ then $\neg x \in \mathcal{F}$

an Aristotelian diagram for $\mathcal{F}$ in $\mathbb{B}$ is a diagram that visualizes an edge-labeled graph $\mathcal{G}$

- the vertices of $\mathcal{G}$ are the elements of $\mathcal{F}$
- the edges of $\mathcal{G}$ are labeled by the relations of $\mathcal{AG}_{\mathbb{B}}$ between those elements
- if $x, y \in \mathcal{F}$ stand in some Aristotelian relation in $\mathbb{B}$, then this is visualized according to the code

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Introduction to Logical Geometry – Part 1
Let $S$ be an appropriate logical system (Boolean + $\models$)

Consider a non-empty fragment $\mathcal{F} \subseteq \mathcal{L}_S$ such that

- every formula $\varphi \in \mathcal{F}$ is $S$-contingent: $\not\models_S \varphi$ and $\not\models_S \neg \varphi$
- $\mathcal{F}$ is closed under negation (up to $\equiv_S$):
  
  if $\varphi \in \mathcal{F}$ then $\exists \psi \in \mathcal{F}: \psi \equiv_S \neg \varphi$

- the formulas in $\mathcal{F}$ are pairwise non-$S$-equivalent:
  
  if $\varphi, \psi \in \mathcal{F}$ are distinct, then $\varphi \not\equiv_S \psi$

An Aristotelian diagram for $\mathcal{F}$ in $S$ is a diagram that visualizes an edge-labeled graph $\mathcal{G}$

- the vertices of $\mathcal{G}$ are the elements of $\mathcal{F}$
- the edges of $\mathcal{G}$ are labeled by the relations of $AG_S$ between those elements
- if $\varphi, \psi \in \mathcal{F}$ stand in some Aristotelian relation in $S$, then this is visualized according to the code
Examples: Aristotelian PCDs for the modal logic S5

100
\( \square p \quad \neg \square p \)

011

1000
\( \square p \quad \neg \square p \)

0111

110
\( \Diamond p \quad \neg \Diamond p \)

001

1100
\( p \quad \neg p \)

0011

- PCD = pair of contradictories
- a PCD is the smallest possible Aristotelian diagram
  - no Aristotelian diagrams with a single formula
  - because of the requirement that they be closed under negation
- PCDs are the building blocks for all larger Aristotelian diagrams
Examples: Aristotelian squares for the modal logic S5

- **classical square**
  - square of opposition

- **degenerate square**
  - unconnectedness square
  - $X$ of opposition

2 PCDs

2 subalternations (SA)

1 contrariety (C)

1 subcontrariety (SC)

2 PCDs

4 $\times$ unconnectedness (U)

Introduction to Logical Geometry – Part 1
Aristotelian hexagons for the modal logic S5

**Jacoby-Sesmat-Blanché hexagon**

- 3 PCDs
- 6 subalternations (SA)
- 3 contrarieties (C)
- 3 subcontrarieties (SC)

**Sherwood-Czezowski hexagon**

- 3 PCDs
- 6 subalternations (SA)
- 3 contrarieties (C)
- 3 subcontrarieties (SC)
Aristotelian octagons for the modal logic S5

Béziau octagon

- 4 PCDs
- 10 SAs & 5 Cs & 5 SCs
- \(4 \times\) unconnectedness (U)

Buridan octagon

- 4 PCDs
- 10 SAs & 5 Cs & 5 SCs
- \(4 \times\) unconnectedness (U)
Boolean closure

- **Boolean closure of a fragment \( F \):**
  - the smallest Boolean algebra that contains \( F \)
  - contains all Boolean combinations of formulas from \( F \)
  - notation: \( B(F) \)
  - contains \( 2^n \) formulas, for some natural number \( n \)

- **Boolean closure of an Aristotelian diagram for \( F \) in \( S \):**
  - Aristotelian diagram for \( B(F) \) in \( S \)
  - note: Aristotelian diagram, so only \( S \)-contingent formulas
  - contains \( 2^n - 2 \) formulas, for some natural number \( n \)

- **some examples:**
  - the Boolean closure of a classical square is a JSB hexagon
    \( \Rightarrow 2^3 - 2 = 6 \) contingent Boolean combinations
  - the Boolean closure of a degenerate square is a rhombic dodecahedron
  - the Boolean closure of an SC hexagon is a rhombic dodecahedron
    \( \Rightarrow 2^4 - 2 = 14 \) contingent Boolean combinations
Boolean closure

- the Boolean closure of a classical square is a JSB hexagon
The effectiveness of bitstring semantics

- logical and diagrammatic effectiveness
- linguistic and cognitive effectiveness:
  - bitstrings generate new questions about
    - the linguistic/cognitive aspects of the expressions they encode
    - the relative weight/strength of individual bit positions inside bitstrings
    - the underlying scalar/linear structure of the conceptual domain
- edges versus center in bitstrings of length 3

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- bitstrings of length 4 as refinements/coarsenings of bitstrings of length 3

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• **no systematic method** for establishing a bitstring semantics for any fragment $\mathcal{F}$ in any logical system $S$

  📜 final part of lecture 1

• no good grasp of the intricate **interplay between Aristotelian and Boolean structure**

  📜 first part of lecture 4

• no good grasp of the **logic-sensitivity of the Aristotelian relations**

  📜 second part of lecture 4

• to overcome these limitations: develop more mathematically precise approach to bitstring semantics
• \( \{0, 1\}^n \) forms a Boolean algebra (bitstrings of length \( n \))
  • \( \land, \lor \) and \( \neg \) are defined componentwise
  • top element: 1\( \cdots \)1
  • bottom element: 0\( \cdots \)0

• we can define the Aristotelian relations between bitstrings:
  two bitstrings \( b_1, b_2 \in \{0, 1\}^n \) are

  - \( n \)-contradictory (\( CD_n \)) iff \( b_1 \land b_2 = 0 \cdots 0 \) and \( b_1 \lor b_2 = 1 \cdots 1 \)
  - \( n \)-contrary (\( C_n \)) iff \( b_1 \land b_2 = 0 \cdots 0 \) and \( b_1 \lor b_2 \neq 1 \cdots 1 \)
  - \( n \)-subcontrary (\( SC_n \)) iff \( b_1 \land b_2 \neq 0 \cdots 0 \) and \( b_1 \lor b_2 = 1 \cdots 1 \)
  - in \( n \)-subalternation (\( SA_n \)) iff \( b_1 \land b_2 = b_1 \) and \( b_1 \lor b_2 \neq b_1 \)

• the Aristotelian geometry for bitstrings of length \( n \):

  \[ \mathcal{A}G_n := \{ CD_n, C_n, SC_n, SA_n \} \]
• the setup:
  • logical systems $S_1, S_2$ and natural numbers $n_1, n_2$
  • $x \in \{S_1, n_1\}$ and $y \in \{S_2, n_2\}$
  • $\mathcal{F}_x$ is a finite set of formulas of system $x$/bitstrings of length $x$
  • $\mathcal{F}_y$ is a finite set of formulas of system $y$/bitstrings of length $y$

• we will define functions from $\mathcal{F}_x$ to $\mathcal{F}_y$

• this encompasses four cases:
  • from formulas of $S_1$ to formulas of $S_2$
  • from formulas of $S_1$ to bitstrings of length $n_2$
  • from bitstrings of length $n_1$ to formulas of $S_2$
  • from bitstrings of length $n_1$ to bitstrings of length $n_2$
the setup:

- logical systems $S_1, S_2$ and natural numbers $n_1, n_2$
- $x \in \{S_1, n_1\}$ and $y \in \{S_2, n_2\}$
- $\mathcal{F}_x$ is a finite set of formulas of system $x$/bitstrings of length $x$
- $\mathcal{F}_y$ is a finite set of formulas of system $y$/bitstrings of length $y$

- a bijection $\gamma : \mathcal{F}_x \to \mathcal{F}_y$ is an Aristotelian isomorphism iff for all Aristotelian relations $R_x \in AG_x$ and corresponding $R_y \in AG_y$, and for all $\varphi, \psi \in \mathcal{F}_x$, it holds that $R_x(\varphi, \psi)$ iff $R_y(\gamma(\varphi), \gamma(\psi))$

- a bijection $\gamma : \mathcal{F}_x \to \mathcal{F}_y$ is a Boolean isomorphism iff there exists some Boolean algebra isomorphism $f : \mathbb{B}(\mathcal{F}_x) \to \mathbb{B}(\mathcal{F}_y)$ such that $\gamma = f \restriction \mathcal{F}_x$

(recall that $\mathbb{B}(\mathcal{F})$ is the Boolean closure of $\mathcal{F}$)
since the Aristotelian relations are defined in purely Boolean terms, the Aristotelian structure of a fragment is entirely determined by its Boolean structure

**lemma**: for any $\gamma: \mathcal{F}_x \rightarrow \mathcal{F}_y$: if $\gamma$ is a Boolean isomorphism, then $\gamma$ is an Aristotelian isomorphism

**a bitstring semantics** for $\mathcal{F}_x$ is a Boolean algebra isomorphism $\beta: \mathbb{B} \rightarrow \{0, 1\}^n$, where $\mathbb{B}$ is some Boolean algebra that contains $\mathcal{F}_x$ (not necessarily the smallest one)

**lemma**: every bitstring semantics $\beta: \mathbb{B} \rightarrow \{0, 1\}^n$ is an Aristotelian isomorphism
Example

- fragment $\mathcal{F}$ of S5-formulas: $\{\Box p, \Diamond p, \Box \neg p, \Diamond \neg p\}$
- two Boolean algebras that contain $\mathcal{F}$:
  - $\mathbb{B}_3$, which has atoms $\Box p, \Diamond p \land \Diamond \neg p$ and $\Box \neg p$ (note: $\mathbb{B}_3 = \mathbb{B}(\mathcal{F})$)
  - $\mathbb{B}_4$, which has atoms $\Box p, p \land \Diamond \neg p, \neg p \land \Diamond p$ and $\Box \neg p$
- two bitstring semantics for $\mathcal{F}$:
  - $\beta_3 : \mathbb{B}_3 \rightarrow \{0, 1\}^3$
  - $\beta_4 : \mathbb{B}_4 \rightarrow \{0, 1\}^4$
Partitions

- let S be a logical system with Boolean operators and a semantics $\models$, and consider $\mathcal{F} = \{\varphi_1, \ldots, \varphi_m\} \subseteq \mathcal{L}_S$

- the **partition of** S induced by $\mathcal{F}$ is
  $$\Pi_S(\mathcal{F}) := \{\alpha \in \mathcal{L}_S \mid \alpha \equiv_S \pm \varphi_1 \land \cdots \land \pm \varphi_m, \text{ and } \alpha \text{ is } S\text{-consistent}\}$$

- $\pm \varphi$ stands for either $\varphi$ or $\neg \varphi$;  $\alpha$ should be read up to $\equiv_S$

- the formulas $\alpha \in \Pi_S(\mathcal{F})$ are called **anchor formulas**
  - in principle, equivalent to a conjunction of $m = |\mathcal{F}|$ conjuncts
  - can often be simplified, e.g. when $\neg \varphi_i \equiv_S \varphi_j$ for some $\varphi_i, \varphi_j \in \mathcal{F}$

- $\Pi_S(\mathcal{F})$ is a **partition** of (the class of all models of) S:
  - $\models_S \neg (\alpha_i \land \alpha_j)$ for distinct $\alpha_i, \alpha_j \in \Pi_S(\mathcal{F})$  (mutually exclusive)
  - $\models_S \bigvee \Pi_S(\mathcal{F})$  (jointly exhaustive)
**Example**

- first-order logic (FOL), fragment $\mathcal{F} := \{\forall xPx, \exists xPx, \neg Pa\}$

- let's compute $\Pi_{FOL}(\mathcal{F})$, the partition of FOL induced by $\mathcal{F}$

- there are $2^{2^3} = 8$ relevant conjunctions

1. $\forall xPx \land \exists xPx \land \neg Pa \leadsto$ FOL-inconsistent
2. $\forall xPx \land \exists xPx \land \neg \neg Pa \leadsto \forall xPx$
3. $\forall xPx \land \neg \exists xPx \land \neg Pa \leadsto$ FOL-inconsistent
4. $\forall xPx \land \neg \exists xPx \land \neg \neg Pa \leadsto$ FOL-inconsistent
5. $\neg \forall xPx \land \exists xPx \land \neg Pa \leadsto \neg Pa \land \exists xPx$
6. $\neg \forall xPx \land \exists xPx \land \neg \neg Pa \leadsto Pa \land \neg \forall xPx$
7. $\neg \forall xPx \land \neg \exists xPx \land \neg Pa \leadsto \neg \exists xPx$
8. $\neg \forall xPx \land \neg \exists xPx \land \neg \neg Pa \leadsto$ FOL-inconsistent

- $\Pi_{FOL}(\mathcal{F}) = \{\forall xPx, \neg Pa \land \exists xPx, Pa \land \neg \forall xPx, \neg \exists xPx\}$
Properties of partitions

- Given partitions \( \Pi_1 \) and \( \Pi_2 \):
  - \( \Pi_1 \) is a **refinement** of \( \Pi_2 \) iff for all \( \alpha \in \Pi_1 \) there exists \( \alpha' \in \Pi_2 \) such that \( \models_S \alpha \rightarrow \alpha' \)
  - The **meet** of \( \Pi_1 \) and \( \Pi_2 \) is defined as follows:
    \[ \Pi_1 \wedge_S \Pi_2 := \{ \gamma_1 \wedge \gamma_2 \mid \gamma_1 \in \Pi_1, \gamma_2 \in \Pi_2, \text{ and } \gamma_1 \wedge \gamma_2 \text{ is } S\text{-consistent} \} \]
  - Note: \( \Pi_1 \wedge_S \Pi_2 \) is the coarsest common refinement of \( \Pi_1 \) and \( \Pi_2 \)

- **Lemma**: if \( \mathcal{F}_1 \cup \mathcal{F}_2 = \mathcal{F} \), then \( \Pi_S(\mathcal{F}_1) \wedge_S \Pi_S(\mathcal{F}_2) = \Pi_S(\mathcal{F}) \)

- **Lemma**: if \( \mathcal{F}_1 \subseteq \mathcal{F}_2 \), then \( \Pi_S(\mathcal{F}_2) \) is a refinement of \( \Pi_S(\mathcal{F}_1) \)

- Given two logics \( S_1 \) and \( S_2 \) (with the same language \( L \)), we say that \( S_2 \) is **stronger** than \( S_1 \) iff for all \( \varphi \in L \): if \( \models_{S_1} \varphi \) then \( \models_{S_2} \varphi \)

- **Lemma**: if \( S_2 \) is stronger than \( S_1 \), then \( \Pi_{S_2}(\mathcal{F}) = \{ \alpha \in \Pi_{S_1}(\mathcal{F}) \mid \alpha \text{ is } S_2\text{-consistent} \} \)
Partition-based bitstring semantics

- logic $S$, fragment $\mathcal{F}$ and partition $\Pi_S(\mathcal{F}) = \{\alpha_1, \ldots, \alpha_n\}$

- **lemma**: for all $\varphi \in \mathbb{B}(\mathcal{F})$:
  - for all $\alpha_i \in \Pi_S(\mathcal{F})$ we have $\models_S \alpha_i \rightarrow \varphi$ or $\models_S \alpha_i \rightarrow \neg \varphi$, but not both
  - $\varphi \equiv_S \bigvee\{\alpha \in \Pi_S(\mathcal{F}) \mid \models_S \alpha \rightarrow \varphi\}$

- for every $\varphi \in \mathbb{B}(\mathcal{F})$, we define a bitstring $\beta^\mathcal{F}_S(\varphi) \in \{0, 1\}^n$ as follows:

  for each bit position $1 \leq i \leq n$: 
  \[[\beta^\mathcal{F}_S(\varphi)]_i := \begin{cases} 
  1 & \text{if } \models_S \alpha_i \rightarrow \varphi, \\
  0 & \text{if } \models_S \alpha_i \rightarrow \neg \varphi. 
  \end{cases} \]

- **lemma**: for all $\varphi \in \mathbb{B}(\mathcal{F})$ we have $\varphi \equiv_S \bigvee\{\alpha_i \in \Pi_S(\mathcal{F}) \mid [\beta^\mathcal{F}_S(\varphi)]_i = 1\}$

- relativized disjunctive normal form: $\varphi$ is rewritten as
  - a disjunction of anchor formulas, which are themselves
  - conjunctions of (possibly negated) formulas $\pm \varphi_j \in \mathcal{F}$
for every $\varphi \in \mathbb{B}(\mathcal{F})$, we have bitstring $\beta^F_S(\varphi) \in \{0, 1\}^n = \{0, 1\}^{\Pi_S(\mathcal{F})}$

- turn this into a function $\beta^F_S : \mathbb{B}(\mathcal{F}) \to \{0, 1\}^{\Pi_S(\mathcal{F})}$

- **theorem:** $\beta^F_S$ is a bitstring semantics for $\mathcal{F}$

- **corollary:** $|\mathbb{B}(\mathcal{F})| = 2^{\Pi_S(\mathcal{F})}$

- **corollary:** $\beta^F_S$ is an Aristotelian isomorphism

- **corollary:** $\beta^F_S$ is a minimal bitstring semantics for $\mathcal{F}$:
  - every other bitstring semantics for $\mathcal{F}$ is either a permutation variant of $\beta^F_S$
  - or makes use of bitstrings of length $> |\Pi_S(\mathcal{F})|$
Correlation between fragment size and bitstring length

- fragment size $m := |\mathcal{F}|$ and bitstring length $n := |\Pi_S(\mathcal{F})|$

- **Theorem:**
  1. We can bound $m$ in terms of $n$: $\lceil \log_2(n) \rceil \leq m \leq 2^n$
  2. We can bound $n$ in terms of $m$: $\lceil \log_2(m) \rceil \leq n \leq 2^m$

- (1) and (2) can be seen as each other's inverses
- All these bounds are tight

- **Theorem** (special case, but very relevant for logical geometry):
  if $\mathcal{F}$ only contains $S$-contingent formulas and is closed under negation:
  1. Bound $m$ in terms of $n$: $2\lceil \log_2(n) \rceil \leq m \leq 2^n - 2$
  2. Bound $n$ in terms of $m$: $\lceil \log_2(m + 2) \rceil \leq n \leq 2^{\frac{m}{2}}$
Thank you!

Questions?

More info: www.logicalgeometry.org