Aristotelian and Boolean Properties of the Keynes-Johnson Octagon of Opposition

Lorenz Demey and Hans Smessaert

(Accepted for publication in *Journal of Philosophical Logic*, 13 May 2024)

Abstract

Around the turn of the 20th century, Keynes and Johnson extended the well-known square of opposition to an octagon of opposition, in order to account for subject negation (e.g., statements like 'all non-S are P'). The main goal of this paper is to study the logical properties of the Keynes-Johnson (KJ) octagons of opposition. In particular, we will discuss three concrete examples of KJ octagons: the original one for subject-negation, a contemporary one from knowledge representation, and a third one (hitherto not yet studied) from deontic logic. We show that these three KJ octagons are all Aristotelian isomorphic, but not all Boolean isomorphic to each other (the first two are representable by bitstrings of length 7, whereas the third one is representable by bitstrings of length 6). These results nicely fit within our ongoing research efforts toward setting up a systematic classification of squares, octagons, and other diagrams of opposition. Finally, obtaining a better theoretical understanding of the KJ octagons allows us to answer some open questions that have arisen in recent applications of these diagrams.

Keywords: square of opposition, octagon of opposition, Aristotelian diagram, J. N. Keynes, W. E. Johnson, bitstring semantics, logical geometry.

1 Introduction

Almost every logician is familiar with the so-called square of opposition. This diagram visualizes the logical relations that obtain between the four categorical statements, which are of the form 'all/some/no/not all S are P'. Around the turn of the 20th century, philosophical logicians such as Keynes (1894) and Johnson (1921) extended the square to an octagon of opposition, in the context of their investigations on subject negation, i.e., statements of the form 'all/some/no/not all non-S are P'. The square of opposition and its various extensions are nowadays called *Aristotelian diagrams*, because of their historical origins in the logical works of Aristotle.[1]

With a few notable exceptions, Keynes and Johnson's octagons did not receive much attention over the course of the 20th century.[2] However, in recent years, this type of diagram has found new heuristic applications in logic-related fields such as natural

[1]The received view holds that Aristotle himself did not draw the actual square diagram, but he did explicitly discuss the categorical statements and some of the logical relations holding between them (Londey and Johanson, 1984). However, see Christensen (2023) for a recent dissenting voice.

[2]This neglect can at least partially be explained in terms of the general climate of hostility between traditional, Aristotelian logic and mathematical, post-Fregean logic, which held sway for most of the 20th century (Demey, 2020a).

language semantics (Dekker, 2015) and knowledge representation (Ciucci et al., 2016; Denœux et al., 2020). This renewed interest is part of a broader trend of studying Aristotelian diagrams with the formal tools and techniques of contemporary logic (Pfeifer and Sanfilippo, 2017; Pizzi, 2016; Rybaříková, 2016), often under the label of *logical geometry* (Demey, 2021; Demey and Smessaert, 2014, 2018a,b; Roelandt, 2016).

The main goal of this paper is to study the logical properties of the Keynes-Johnson octagons of opposition. In particular, we will discuss three concrete examples of this type of octagon, and show that they are all Aristotelian isomorphic, but not all Boolean isomorphic to each other. (The notions of Aristotelian isomorphism and Boolean isomorphism will be formally defined later in the paper.) These results nicely fit within the ongoing effort in logical geometry toward setting up a systematic classification of Aristotelian families and their Boolean subfamilies.[3] Finally, obtaining a better theoretical understanding of the Keynes-Johnson octagons and the notions of Aristotelian/Boolean isomorphism will allow us to answer some open questions and to clear up some misunderstandings that have arisen over the past decade.

The paper is organized as follows. In Section 2 we present Keynes (1894) and Johnson (1921)'s original octagon for subject negation, discuss its historical development, and analyze its Boolean properties by means of bitstring semantics. Next, in Section 3, we present a second Keynes-Johnson octagon, which has recently been discussed extensively in the field of knowledge representation. We show that this second octagon is Aristotelian as well as Boolean isomorphic to the first one. In Section 4 we then present a third Keynes-Johnson octagon, which arises naturally in the context of deontic logic. Crucially, we show that this third octagon is Aristotelian isomorphic, but *not* Boolean isomorphic, to the first two. Section 5 puts the results of the previous three sections in a broader theoretical context. In particular, we show that the Aristotelian family of Keynes-Johnson (KJ) octagons has precisely two Boolean subfamilies, viz., one subfamily of KJ octagons that are representable by bitstrings of length 7 and one subfamily of KJ octagons that are representable by bitstrings of length 6. Finally, Section 6 wraps things up, and mentions some potential directions for future research. It should also be noted that throughout the paper, we have included some brief discussions of notions and results that are defined/proved in far more detail elsewhere, which should help to keep the paper relatively self-contained.

2 Keynes and Johnson's Octagon for Subject Negation

In this section we will present Keynes (1894) and Johnson (1921)'s original octagon of opposition, discuss its historical development, and analyze its Boolean properties by means of bitstring semantics. However, in order to facilitate the comparison of this diagram with other ones later in the paper, it will be useful to first define the Aristotelian relations in a more precise and abstract fashion than is usually done. In general, these relations can be defined with respect to an arbitrary Boolean algebra (Demey, 2019b; Demey and Smessaert, 2016b):

Definition 1. Let $\mathbb{B} = \langle B, \wedge_{\mathbb{B}}, \vee_{\mathbb{B}}, \neg_{\mathbb{B}}, \top_{\mathbb{B}}, \bot_{\mathbb{B}} \rangle$ be a Boolean algebra. Two elements $x, y \in B$ are said to be

[3] When setting up such a classification, we typically restrict ourselves to diagrams that only contain contingent formulas and that are closed under negation. Such diagrams always have an even number $2n$ of formulas, and can also be viewed as consisting of n *pairs of contradictory formulas* (PCDs). A diagram consisting of n PCDs is called a σ_n-diagram; for example, σ_2-, σ_3- and σ_4-diagrams are squares, hexagons and octagons of opposition, respectively (Demey and Smessaert, 2016a; De Klerck et al., 2023).

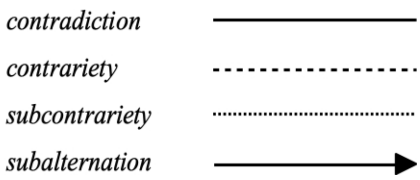

Figure 1: Code for visualizing the Aristotelian relations.

$\mathbb{B}$-*contradictory* ($CD_{\mathbb{B}}$)	iff	$x \wedge_{\mathbb{B}} y = \bot_{\mathbb{B}}$	and	$x \vee_{\mathbb{B}} y = \top_{\mathbb{B}}$,
$\mathbb{B}$-*contrary* ($C_{\mathbb{B}}$)	iff	$x \wedge_{\mathbb{B}} y = \bot_{\mathbb{B}}$	and	$x \vee_{\mathbb{B}} y \neq \top_{\mathbb{B}}$,
$\mathbb{B}$-*subcontrary* ($SC_{\mathbb{B}}$)	iff	$x \wedge_{\mathbb{B}} y \neq \bot_{\mathbb{B}}$	and	$x \vee_{\mathbb{B}} y = \top_{\mathbb{B}}$,
in $\mathbb{B}$-*subalternation* ($SA_{\mathbb{B}}$)	iff	$x \wedge_{\mathbb{B}} y = x$	and	$x \vee_{\mathbb{B}} y \neq x$.

The *Aristotelian geometry for* $\mathbb{B}$ is the set consisting of the above four relations, i.e., $\mathcal{AG}_{\mathbb{B}} := \{CD_{\mathbb{B}}, C_{\mathbb{B}}, SC_{\mathbb{B}}, SA_{\mathbb{B}}\}$. Finally, x and y are said to be $\mathbb{B}$-*unconnected* iff they do not stand in any of the above relations, i.e., iff (i) $x \wedge_{\mathbb{B}} y \neq \bot_{\mathbb{B}}$, (ii) $x \vee_{\mathbb{B}} y \neq \top_{\mathbb{B}}$, (iii) $x \wedge_{\mathbb{B}} y \neq x$ and (iv) $x \wedge_{\mathbb{B}} y \neq y$. The Aristotelian relations are visualized following the convention shown in Figure 1.[4]

This definition can be seen as an abstract 'template': more concrete characterizations of the Aristotelian relations can be obtained by plugging in concrete Boolean algebras for $\mathbb{B}$. For example, by taking $\mathbb{B}$ to be the powerset $\wp(X)$ of some set X, we can say that two sets $A, B \subseteq X$ are $\wp(X)$-contrary iff $A \cap B = \emptyset$ and $A \cup B \neq X$. This definition also subsumes the usual, informal characterization of the Aristotelian relations. After all, if S is a standard logical system with a model-theoretic semantics $\models_{\mathsf{S}}$ and a language $\mathcal{L}_{\mathsf{S}}$ that includes the Boolean connectives, then its Lindenbaum-Tarski algebra $\mathbb{B}(\mathsf{S}) := \mathcal{L}_{\mathsf{S}}/{\equiv_{\mathsf{S}}} = \{[\varphi]_{\mathsf{S}} \mid \varphi \in \mathcal{L}_{\mathsf{S}}\}$ (where $[\varphi]_{\mathsf{S}} := \{\psi \in \mathcal{L}_{\mathsf{S}} \mid \varphi \equiv_{\mathsf{S}} \psi\}$) constitutes a Boolean algebra, and can thus be plugged in for $\mathbb{B}$ in Definition 1. It is easy to see that this corresponds precisely to the informal characterization of the Aristotelian relations; for example, given any two formulas $\varphi, \psi \in \mathcal{L}_{\mathsf{S}}$, it holds that

	$[\varphi]_{\mathsf{S}}$ and $[\psi]_{\mathsf{S}}$ are $\mathbb{B}(\mathsf{S})$-contrary		
iff	$[\varphi]_{\mathsf{S}} \wedge [\psi]_{\mathsf{S}} = \bot$	and	$[\varphi]_{\mathsf{S}} \vee [\psi]_{\mathsf{S}} \neq \top$
iff	$[\varphi \wedge \psi]_{\mathsf{S}} = \bot$	and	$[\varphi \vee \psi]_{\mathsf{S}} \neq \top$
iff	$\models_{\mathsf{S}} \neg(\varphi \wedge \psi)$	and	$\not\models_{\mathsf{S}} \varphi \vee \psi$
iff	φ and ψ cannot be true together	and	φ and ψ can be false together.

When there is no danger of confusion, we will not draw a sharp distinction between the logical system S and its Lindenbaum-Tarski algebra $\mathbb{B}(\mathsf{S})$. For example, we will simply say that φ and ψ are S-contrary, rather than that $[\varphi]_{\mathsf{S}}$ and $[\psi]_{\mathsf{S}}$ are $\mathbb{B}(\mathsf{S})$-contrary. (Note that the Aristotelian relations are thus defined up to logical equivalence; for example, if $\varphi \equiv_{\mathsf{S}} \varphi'$ and $\psi \equiv_{\mathsf{S}} \psi'$, then φ and ψ are S-contrary iff φ' and ψ' are S-contrary.)

With Definition 1 in place, we now turn to Keynes (1894) and Johnson (1921)'s octagon of opposition. This octagon visualizes the categorical statements with subject negation. These eight statements are listed below, together with their formalizations in the language of first-order logic.[5]

[4] Note that unconnectedness is not visualized at all. After all, unconnectedness corresponds to the absence of any Aristotelian relation, and is thus naturally 'visualized' by the absence of any visual element.

[5] Note that these first-order formalizations completely sidestep the issues of existential import that surround the categorical statements; we will return to this in Footnote 8.

Definition 2. The eight propositions that appear in the Keynes-Johnson octagon for subject negation:

1.	all S are P	$\forall x(Sx \rightarrow Px)$
2.	some S are P	$\exists x(Sx \wedge Px)$
3.	no S are P	$\forall x(Sx \rightarrow \neg Px)$
4.	not all S are P	$\exists x(Sx \wedge \neg Px)$
5.	all non-S are P	$\forall x(\neg Sx \rightarrow Px)$
6.	some non-S are P	$\exists x(\neg Sx \wedge Px)$
7.	no non-S are P	$\forall x(\neg Sx \rightarrow \neg Px)$
8.	not all non-S are P	$\exists x(\neg Sx \wedge \neg Px)$

The set consisting of these eight propositions will be denoted $\mathcal{F}_{sn}$.

These statements have a long history in logic. In his *De interpretatione*, Aristotle already examined the various opposition relations that obtain between sentences such as‘every man walks’, ‘every man does not walk’, ‘every not-man walks’, ‘every not-man does not walk’, etc. (Wilkinson Miller, 1938; Ackrill, 1963; Jones, 2010). These investigations were continued by Apuleius and Boethius (Alvarez and Correia, 2017; Correia, 2009, 2017), and in medieval logic, subject negation was discussed under the heading of ‘infinitizing negation’ (Parsons, 2008, 2014). The topic was also studied by Arabic logicians such as Avicenna (Hodges, 2018), and in the 20th century, it continued to gather the interest of renowned logicians such as Haskell B. Curry (1936) and Arthur N. Prior (1955).

In order to visualize the logical relations holding between the eight statements of $\mathcal{F}_{sn}$, one needs an octagon of opposition, as shown in Figure 2. A version of this diagram first appeared in the third edition of John N. Keynes’ *Studies and Exercises in Formal Logic* (Keynes, 1894, p. 113).[6] Keynes acknowledged the help of William E. Johnson in setting up this octagon, and the same diagram also appeared in the latter’s textbook (Johnson, 1921, p. 142). Throughout the 20th century, this octagon has been discussed — albeit sometimes in very different shapes, e.g., as a three-dimensional cube instead of a two-dimensional octagon — by authors such as de Laguna (1912), Dopp (1949, 1960), Thomas (1949), Grosjean (1972), Hacker (1975) and Sauriol (1976). In more recent years, it has been studied by philosophers and logicians such as Moktefi and Shin (2012), Dekker (2015), Demey (2015), García-Cruz (2017) and Moktefi and Schang (2023), but it has also received ample attention from mathematicians and computer scientists such as Libert (2012), Ciucci et al. (2012, 2015), Dubois and Prade (2012, 2015a) and Dubois et al. (2015a,b, 2017a,b).

The Keynes-Johnson octagon for subject negation is to be interpreted on the assumption that the subject and predicate terms are neither empty nor exhaustive of the entire universe. Most authors were very explicit about this assumption. For example, Johnson (1921, pp. 139–140) wrote: “In what follows we shall adopt the traditional view that there are instances of S, of non-S, of P, and of non-P; and on this assumption we proceed to consider all the formal relations amongst the propositions involving S or non-S with P or non-P” (notation adapted to match that of the present paper). Formally, this means that the eight formulas from $\mathcal{F}_{sn}$ are interpreted relative to the

[6] The first (1884) and second (1887) editions of this work contain the traditional square, but not yet the octagon. The last major edition of this work, viz., the fourth (1906), also contains the octagon. Note that for reasons of space, we are here focusing exclusively on *published* materials. Recent archival work has shown that some of Augustus De Morgan’s handwritten notes (dated February 1853) contain several versions of this octagon as well (Heinemann and Demey, 2022).

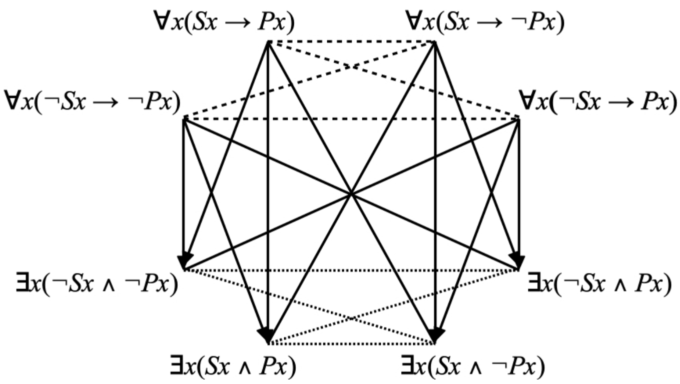

Figure 2: Octagon of opposition for $(\mathcal{F}_{sn}, \mathsf{EFOL})$.

logical system EFOL (for 'existential first-order logic'), which has the same language $\mathcal{L}_{\mathsf{FOL}}$ as ordinary FOL, but is axiomatized by adding $\exists x Fx$ and $\exists x \neg Fx$ (for any unary predicate symbol F) as additional axioms to FOL. This logical system is naturally interpreted on first-order models $\langle D, I\rangle$ (with domain D and interpretation function I) that satisfy the additional requirement that $\emptyset \neq I(F) \neq D$ for all unary predicate symbols F.[7,8] Its Lindenbaum-Tarski algebra $\mathbb{B}(\mathsf{EFOL})$ is a Boolean algebra, and can thus be plugged into Definition 1. In this way we can characterize the Aristotelian relations that obtain between the categorical statements with subject negation, as shown in the Keynes-Johnson octagon in Figure 2. We briefly discuss two examples:

- $\forall x(Sx \to Px)$ and $\forall x(Sx \to \neg Px)$ are EFOL-contrary

  Suppose, toward a contradiction, that these formulas are both satisfied by an EFOL-model $\langle D, I\rangle$. Then it would follow that $I(S) = \emptyset$, which is ruled out by the definition of EFOL-model. Hence, $\models_{\mathsf{EFOL}} \neg(\forall x(Sx \to Px) \wedge \forall x(Sx \to \neg Px))$. On the other hand, it is easy to see that there does exist an EFOL-model $\langle D, I\rangle$ in which these formulas are both false; for example, one can take

[7] The system EFOL has also been called SYL (for 'syllogistics'), cf. Demey and Smessaert (2018b) and Smessaert and Demey (2023), as well as $\mathsf{FOL}_{\exists}$, which is intertranslatable with QUARC, cf. Ben-Yami (2014, forthcoming) and Raab (2016). More importantly, note that EFOL is not closed under uniform substitution: for example, $\exists x Fx$ is an EFOL-tautology but $\exists x(Fx \wedge \neg Fx)$ is not. Since the principle of uniform substitution is often viewed as the technical counterpart of the philosophical idea that logic is a purely *formal* enterprise, this failure of uniform substitution might be seen as not merely a technical property of EFOL, but rather a fundamental philosophical objection against this logical system. However, there are many other well-known logics that fail to be closed under uniform substitution, such as Carnap's (1947) modal logic C, data logic DL (Veltman, 1985), public announcement logic PAL (van Ditmarsch et al., 2007; Holliday et al., 2013) and inquisitive logic InqL (Ciardelli et al., 2019). Recently, Punčochář (2023) has argued that failure of uniform substitution does *not* undermine the philosophical idea that logic is a purely formal enterprise. Rather, there is a much weaker technical principle, called *closure under syntactically isomorphic substitution* (Schurz, 2001), which captures this philosophical idea more closely, and which does hold for all of the aforementioned logical systems (including EFOL).

[8] Note that we take 'existential import' to be a property of the underlying *logical system* (cf. the additional axioms of EFOL and the additional requirements on EFOL-models). Alternatively, we could have interpreted it as a property of *individual formulas*, which would mean that we continue to work in ordinary FOL, and add the required assumptions as conjuncts to the formulas. For example, the formalization of 'all S is P' would then have to be $\forall x(Sx \to Px) \wedge \exists x Sx \wedge \exists x \neg Px$. (Note that $\exists x \neg Sx$ and $\exists x Px$ do not have to be added as conjuncts, because they are already entailed by the conjunction as stated.) Both of these methods of formalizing the intuitive notion of existential import have their advantages and disadvantages, but in this paper we have chosen to view it as a property of the logical system, because this seems to remain closest to Keynes and Johnson's original intentions (cf. the quotation given above).

$D := \{a, b, c\}$, $I(S) := \{a, b\}$ and $I(P) := \{a\}$. Hence, $\not\models_{\mathsf{EFOL}} \forall x(Sx \to Px) \vee \forall x(Sx \to \neg Px)$.

- $\forall x(Sx \to Px)$ and $\forall x(\neg Sx \to Px)$ are EFOL-contrary

  Suppose, toward a contradiction, that these formulas are both satisfied by an EFOL-model $\langle D, I\rangle$. Then it would follow that $I(P) = D$, which is ruled out by the definition of EFOL-model. Hence, $\models_{\mathsf{EFOL}} \neg(\forall x(Sx \to Px) \wedge \forall x(\neg Sx \to Px))$. On the other hand, it is again easy to see that there does exist an EFOL-model in which these formulas are both false, and hence $\not\models_{\mathsf{EFOL}} \forall x(Sx \to Px) \vee \forall x(\neg Sx \to Px)$.

It should be emphasized that these Aristotelian relations obtain *relative to the logical system* EFOL. If we move to another logical system, we might lose or gain some relations. On the one hand, we can move from EFOL to the weaker system of FOL. In that case, we would lose all Aristotelian relations, except for the four contradiction relations. For example, $\forall x(Sx \to Px)$ and $\forall x(Sx \to \neg Px)$ are EFOL-contrary, but FOL-unconnected.[9] On the other hand, we can move from EFOL to the stronger system REFOL, which was first studied by Reichenbach (1952). This system contains two more axioms, viz., $\exists x(Sx \leftrightarrow Px)$ and $\exists x(Sx \leftrightarrow \neg Px)$, and requires the additional conditions on first-order models $\langle D, I\rangle$ that $I(S) \neq D \backslash I(P)$ and $I(S) \neq I(P)$.[10] In this case, we would gain some additional Aristotelian relations. For example, $\forall x(Sx \to Px)$ and $\forall x(\neg Sx \to \neg Px)$ are EFOL-unconnected, but REFOL-contrary.[11] These considerations show that the Keynes-Johnson octagon for subject negation is highly sensitive to the details of the underlying logical system. This logic-sensitivity of Aristotelian diagrams is a well-known phenomenon in logical geometry (Demey, 2015; Demey and Frijters, 2023).

We will now investigate the Boolean properties of the Keynes-Johnson octagon for subject negation, by computing its *bitstring semantics*. Bitstrings are combinatorial representations of formulas that provide a concrete grip on the logical behavior of a given Aristotelian diagram. Demey and Smessaert (2018b) describe in detail how to compute the bitstring semantics of any Aristotelian diagram in the context of logical systems; we will here briefly describe the key steps of this technique in the context of Boolean algebras.

Definition 3. Given a fragment $\mathcal{F} = \{x_1, \ldots, x_m\}$ of some Boolean algebra $\mathbb{B}$, the *partition of* $\mathbb{B}$ *induced by* $\mathcal{F}$ is defined as

$$\Pi_{\mathbb{B}}(\mathcal{F}) := \{a \in B \mid a = \pm x_1 \wedge_{\mathbb{B}} \cdots \wedge_{\mathbb{B}} \pm x_m, \text{ and } a \neq \bot_{\mathbb{B}}\}$$

(where $+x = x$ and $-x = \neg_{\mathbb{B}} x$).[12] Furthermore, the *Boolean closure of* $\mathcal{F}$ *in* $\mathbb{B}$, denoted $\mathbb{B}(\mathcal{F})$, is the smallest Boolean subalgebra of $\mathbb{B}$ that contains $\mathcal{F}$.

[9] In particular, these two formulas can be true together in FOL, as they are both satisfied by any first-order model $\langle D, I\rangle$ which has $I(S) = \emptyset$. Note that such structures are models of FOL, but not of EFOL.

[10] Reichenbach (1952, pp. 3–4) himself describes this as follows: "none of the four classes S, P, $\overline{S}$, $\overline{P}$, is empty and *no two of them are identical*" (emphasis added).

[11] The presence of these additional Aristotelian relations means that Reichenbach (1952)'s diagram in REFOL is fundamentally different from Keynes (1894) and Johnson (1921)'s diagram in EFOL; more precisely, these two diagrams are *not Aristotelian isomorphic* to each other (cf. Definition 6). It is thus incorrect to simply identify these two diagrams with one another, as is done by Ciucci et al. (2015, 2016) and Dubois et al. (2017a,b). We will briefly revisit Reichenbach's diagram in Footnote 29.

[12] The set $\Pi_{\mathbb{B}}(\mathcal{F})$ is called a 'partition' of $\mathbb{B}$ because its elements are (i) jointly exhaustive, i.e., $\bigvee_{\mathbb{B}} \Pi_{\mathbb{B}}(\mathcal{F}) = \top_{\mathbb{B}}$, and (ii) mutually exclusive, i.e., $a \wedge_{\mathbb{B}} a' = \bot_{\mathbb{B}}$ for distinct $a, a' \in \Pi_{\mathbb{B}}(\mathcal{F})$.

Informally, $\mathbb{B}(\mathcal{F})$ contains the Boolean combinations of elements from $\mathcal{F}$, and nothing else. It can be shown that every element in the Boolean closure of $\mathcal{F}$ is a join of elements of the partition induced by $\mathcal{F}$: for every $y \in \mathbb{B}(\mathcal{F})$ we have

$$y \quad = \quad \bigvee_{\mathbb{B}} \{a \in \Pi_{\mathbb{B}}(\mathcal{F}) \mid \ a \leq_{\mathbb{B}} y\}$$

(where $a \leq_{\mathbb{B}} y$ means that $a \wedge_{\mathbb{B}} y = a$, as usual). The notation format of bitstrings is then introduced to 'keep track' of which elements of $\Pi_{\mathbb{B}}(\mathcal{F})$ enter into this join:

Definition 4. Given a fragment $\mathcal{F}$ of some Boolean algebra $\mathbb{B}$, as in Definition 3, the *bitstring semantics* $\beta^{\mathcal{F}}_{\mathbb{B}} \colon \mathbb{B}(\mathcal{F}) \to \{0,1\}^{|\Pi_{\mathbb{B}}(\mathcal{F})|}$ maps each element $y \in \mathbb{B}(\mathcal{F})$ to its bitstring representation $\beta^{\mathcal{F}}_{\mathbb{B}}(y) \in \{0,1\}^{|\Pi_{\mathbb{B}}(\mathcal{F})|}$. Concretely, we fix an ordering $a_1, \dots, a_{|\Pi_{\mathbb{B}}(\mathcal{F})|}$ of the partition $\Pi_{\mathbb{B}}(\mathcal{F})$, and for each $1 \leq i \leq |\Pi_{\mathbb{B}}(\mathcal{F})|$, we define the i^{th} bit position as follows:

$$[\beta^{\mathcal{F}}_{\mathbb{B}}(y)]_i := \begin{cases} 1 & \text{if } a_i \leq_{\mathbb{B}} y \\ 0 & \text{otherwise.} \end{cases}$$

For example, if $\Pi_{\mathbb{B}}(\mathcal{F}) = \{a_1, a_2, a_3, a_4\}$ and for some $y \in \mathbb{B}(\mathcal{F})$ it holds that $a_1 \leq_{\mathbb{B}} y$ and $a_3 \leq_{\mathbb{B}} y$, then we have $y = a_1 \vee_{\mathbb{B}} a_3$ and write $\beta^{\mathcal{F}}_{\mathbb{B}}(y) = 1010$. Note that $|\Pi_{\mathbb{B}}(\mathcal{F})|$ is the *length* of the bitstring $\beta^{\mathcal{F}}_{\mathbb{B}}(y)$. It can be shown that $\beta^{\mathcal{F}}_{\mathbb{B}}$ is a Boolean algebra isomorphism; hence, the Boolean closure $\mathbb{B}(\mathcal{F})$ of $\mathcal{F}$ contains exactly $2^{|\Pi_{\mathsf{S}}(\mathcal{F})|}$ elements. The bitstring length $|\Pi_{\mathbb{B}}(\mathcal{F})|$ thus provides a direct measure of the Boolean complexity of $\mathcal{F}$.

After this quick summary of bitstring semantics, we will now apply this technique to the Keynes-Johnson octagon for subject negation. Recall that $\mathcal{F}_{sn}$ is the set of eight propositions that appear in this octagon (cf. Definition 2). As we have seen above, these formulas come from the logical system EFOL, i.e., we have $\mathcal{F}_{sn} \subseteq \mathbb{B}(\mathsf{EFOL})$. In order to determine the partition of $\mathbb{B}(\mathsf{EFOL})$ that is induced by $\mathcal{F}_{sn}$, we have to consider all conjunctions of (possibly negated) propositions from $\mathcal{F}_{sn}$. Many of these conjunctions will turn out to be EFOL-inconsistent, and can thus be discarded. For example, any conjunction that simultaneously contains $\forall x(Sx \to Px)$ and $\forall x(Sx \to \neg Px)$ as conjuncts is EFOL-inconsistent, because those two formulas are EFOL-contrary. By systematically going through all conjunctions of this form, rewriting the conjunctions as simpler, EFOL-equivalent propositions whenever possible, and discarding the EFOL-inconsistent conjunctions, we find the partition induced by the Keynes-Johnson octagon for subject negation:

$$\begin{array}{lclcll}
\Pi_{\mathsf{EFOL}}(\mathcal{F}_{sn}) & = & \{ & \alpha_1 & := & \forall x(Sx \to Px) \wedge \forall x(\neg Sx \to \neg Px), \\
& & & \alpha_2 & := & \forall x(Sx \to Px) \wedge \exists x(\neg Sx \wedge Px), \\
& & & \alpha_3 & := & \forall x(Sx \to \neg Px) \wedge \forall x(\neg Sx \to Px), \\
& & & \alpha_4 & := & \forall x(Sx \to \neg Px) \wedge \exists x(\neg Sx \wedge \neg Px), \\
& & & \alpha_5 & := & \exists x(Sx \wedge \neg Px) \wedge \forall x(\neg Sx \to \neg Px), \\
& & & \alpha_6 & := & \exists x(Sx \wedge Px) \wedge \forall x(\neg Sx \to Px), \\
& & & \alpha_7 & := & \exists x(Sx \wedge Px) \wedge \exists x(Sx \wedge \neg Px) \wedge \\
& & & & & \exists x(\neg Sx \wedge Px) \wedge \exists x(\neg Sx \wedge \neg Px) \qquad \}.
\end{array}$$

For ease of notation, we will write the bitstring semantics $\beta^{\mathcal{F}_{sn}}_{\mathsf{EFOL}}$ that corresponds to this partition simply as β_{sn}. Since $|\Pi_{\mathsf{EFOL}}(\mathcal{F}_{sn})| = 7$, the Keynes-Johnson octagon for subject negation can be represented by means of bitstrings of length 7. For example, since $\forall x(Sx \to Px)$ is EFOL-equivalent to $\alpha_1 \vee \alpha_2$, it can be represented

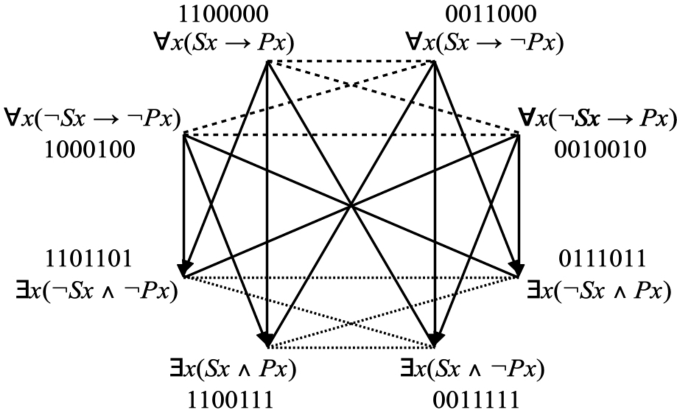

Figure 3: Bitstring semantics β_{sn} for the octagon of opposition for $(\mathcal{F}_{sn}, \mathsf{EFOL})$.

as the bitstring 1100000, i.e., $\beta_{sn}(\forall x(Sx \to Px)) = 1100000$. The bitstrings of all eight propositions in the Keynes-Johnson octagon for subject negation are shown in Figure 3.[13] Finally, since $|\Pi_{\mathsf{EFOL}}(\mathcal{F}_{sn})| = 7$, it follows that the Boolean closure of the Keynes-Johnson octagon for subject negation is isomorphic to $\{0,1\}^7$, i.e., $\mathbb{B}_{\mathsf{EFOL}}(\mathcal{F}_{sn}) \cong \{0,1\}^7$. This Boolean closure thus contains $2^7 = 128$ elements, i.e., there exist 128 distinct Boolean combinations of the categorical statements with subject negation.

To conclude this section, we note that the seven elements of $\Pi_{\mathsf{EFOL}}(\mathcal{F}_{sn})$ describe the seven relations that can obtain between two *sets* S and P — on the assumption that $\emptyset \neq S, P \neq D$. This connection was already discussed in detail by Keynes (1894, pp. 140–144) and Johnson (1921, pp. 144–155).[14] Furthermore, if the sets S and P are viewed as sets of possible worlds, i.e., as *propositions*, then the seven elements of $\Pi_{\mathsf{EFOL}}(\mathcal{F}_{sn})$ also describe the seven relations that can obtain between two propositions φ and ψ — on the assumption that φ and ψ are contingent.[15] In Smessaert and Demey (2014) it is shown that these seven relations naturally cluster together into (i) the *opposition relations* of contradiction (α_3), contrariety (α_4) and subcontrariety (α_6), (ii) the *implication relations* of bi-implication (α_1), left-implication (α_2) and right-implication (α_5), and finally, (iii) unconnectedness (α_7), which indicates the absence of any opposition or implication relation. Furthermore, these seven relations can be ordered according to their informativity, with contradiction and bi-implication being the most informative relations and unconnectedness being the least informative.[16]

[13] Also cf. Moktefi and Schang (2023, Figure 11).

[14] Also cf. Moktefi and Schang (2023, Figure 3).

[15] This was already hinted at by Keynes in the fourth edition of his *Studies and Exercises in Formal Logic*: "These seven possible relations between propositions (taken in pairs) will be found to be precisely analogous to the seven possible relations between classes (taken in pairs)" (1906, p. 119), and: "this sevenfold scheme of class relations should be compared with the sevenfold scheme of relations between propositions" (1906, p. 174).

[16] This informativity ordering is discussed in detail by Smessaert and Demey (2014). The same ordering is also obtained by Alvarez-Fontecilla (2016), whose notion of the 'strength' of a logical relation corresponds to our notion of 'informativity'. For further discussion about the seven relations between sets/propositions, see Doyle (1952), Prior (1953), Menne (1954), Rose (1957), Czeżowski (1958), Baker (1977), Kennedy (1985), Furs (1987), Johnson (1997), Humberstone (2013) and Łukasiewicz (2017), as well as the textbooks by Cohen and Nagel (1930), Stebbing (1930, 1934), Lemmon (1965) and Hamblin (1967).

3 An Octagon for Knowledge Representation

In this section we will analyze a second Keynes-Johnson octagon, which has recently been used extensively in the heuristics of knowledge representation (Amgoud and Prade, 2013; Ciucci et al., 2014, 2016; Dubois and Prade, 2015a,b; Dubois et al., 2015b; Denœux et al., 2020).[17] This octagon is based on a binary relation $R \subseteq X \times Y$ and a subset $S \subseteq Y$. With these ingredients, one can define the set $R(S) := \{x \in X \mid \exists s \in S\colon xRs\}$. By applying set-theoretical complementation to the set S (relative to Y), to the relation R (relative to $X \times Y$) or to the resulting set $R(S)$ (relative to X), we obtain a total number of eight subsets of X.

Definition 5. The eight sets that appear in the Keynes-Johnson octagon for knowledge representation:

$$
\begin{array}{rlcl@{\quad}c@{\quad}l}
1. & R(S) & = & \{x \in X \mid \exists s \in S\colon xRs\} & & \\
2. & R(\overline{S}) & = & \{x \in X \mid \exists s \in \overline{S}\colon xRs\} & & \\
3. & \overline{R}(S) & = & \{x \in X \mid \exists s \in S\colon \neg\, xRs\} & & \\
4. & \overline{R}(\overline{S}) & = & \{x \in X \mid \exists s \in \overline{S}\colon \neg\, xRs\} & & \\
5. & \overline{R(S)} & = & \{x \in X \mid \neg\exists s \in S\colon xRs\} & = & \{x \in X \mid \forall s \in S\colon \neg xRs\} \\
6. & \overline{R(\overline{S})} & = & \{x \in X \mid \neg\exists s \in \overline{S}\colon xRs\} & = & \{x \in X \mid \forall s \in \overline{S}\colon \neg xRs\} \\
7. & \overline{\overline{R}(S)} & = & \{x \in X \mid \neg\exists s \in S\colon \neg\, xRs\} & = & \{x \in X \mid \forall s \in S\colon xRs\} \\
8. & \overline{\overline{R}(\overline{S})} & = & \{x \in X \mid \neg\exists s \in \overline{S}\colon \neg\, xRs\} & = & \{x \in X \mid \forall s \in \overline{S}\colon xRs\}
\end{array}
$$

The set consisting of these eight sets will be denoted $\mathcal{F}_{kr}$.

Depending on the specific interpretation that one assigns to X, Y and R, one obtains various concrete knowledge representation formalisms. For example, if we interpret X as a set of objects, Y as a set of properties, and R as expressing which objects have which properties, then we are working in *formal concept analysis*. By contrast, if we assume that $X = Y$ and interpret both as a set of arguments, and take R to express the attack relation between arguments, then we are in *formal argumentation theory*. Taking $X = Y$ to be a set of objects and R to be an indiscernibility relation on objects leads us to the formalism of *rough set theory*. Finally, if we interpret $X = Y$ as a set of possible worlds and R as an accessibility relation on possible worlds, we are working in *modal logic*; for example, in a given Kripke model $\mathbb{M} = \langle X, R, V\rangle$, we have $[\![\Diamond p]\!]^{\mathbb{M}} = R([\![p]\!]^{\mathbb{M}})$.[18] Since this relational perspective underlies such a wide variety of knowledge representation formalisms, the diagrams it gives rise to can help to explore fruitful parallels between different formalisms, establish new bridges, and investigate new notions by transferring them from one formalism to another (Ciucci et al., 2014; Dubois et al., 2015b; Denœux et al., 2020). Furthermore, this heuristic role does not have to remain limited to specific applications within knowledge representation, since relationally-inspired diagrams also allow us to bridge entire research areas, such as artificial intelligence, linguistics and philosophy (Marquis et al., 2020).

The Keynes-Johnson octagon for knowledge representation is to be interpreted on the assumption that the relations R and $\overline{R}$ are both serial (i.e., for all $x \in X$, there exist $y, y' \in Y$ such that xRy and not xRy') and that the set $S \subseteq Y$ is non-trivial, i.e., $\emptyset \neq$

[17] In most of these papers, the diagram is shown as a three-dimensional cube instead of a two-dimensional octagon, but this difference in visualization is irrelevant from a logical perspective.

[18] As usual, $[\![\varphi]\!]^{\mathbb{M}}$ denotes the truth set of the formula φ in $\mathbb{M}$, i.e., $[\![\varphi]\!]^{\mathbb{M}} := \{x \in X \mid \mathbb{M}, x \models \varphi\}$.

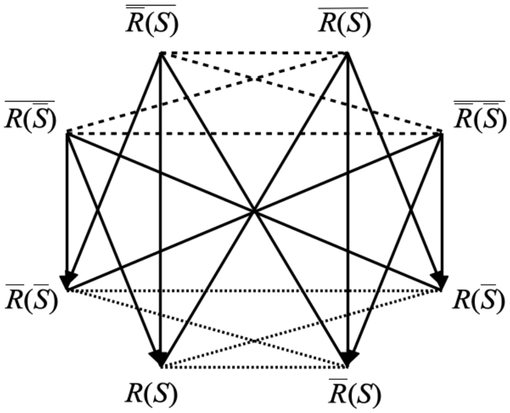

Figure 4: Octagon of opposition for $(\mathcal{F}_{kr}, \wp(X))$.

$S \neq Y$ (Ciucci et al., 2016, p. 355).[19] The eight sets given in Definition 5 are subsets of X, and thus elements of the Boolean algebra $\wp(X)$. By means of Definition 1 we can thus characterize the Aristotelian relations that obtain between these eight sets, as shown in the Keynes-Johnson octagon in Figure 4. For example, in order to see that $\overline{\overline{R}(S)}$ and $\overline{R(S)}$ are $\wp(X)$-contrary, we have to check the following two conditions:

- $\overline{\overline{R}(S)} \cap \overline{R(S)} = \emptyset$

  Suppose, toward a contradiction, that there is some $x \in \overline{\overline{R}(S)} \cap \overline{R(S)}$. It would follow that for all $s \in S$, both xRs and not xRs. This means that $S = \emptyset$, which contradicts our assumption that $S \neq \emptyset$.

- $\overline{\overline{R}(S)} \cup \overline{R(S)} \neq X$

  Take any $x \in X$ such that $\exists s \in S\colon xRs$ and $\exists s \in S\colon \neg xRs$ (the existence of such an x is not ruled out by our assumptions that R and $\overline{R}$ are serial and that $\emptyset \neq S \neq Y$). Hence $x \in \overline{R}(S) \cap R(S)$ and thus $x \notin \overline{\overline{R}(S) \cap R(S)} = \overline{\overline{R}(S)} \cup \overline{R(S)}$. This shows that $\overline{\overline{R}(S)} \cup \overline{R(S)} \neq X$.

Upon visual inspection, it becomes immediately clear that the octagon for subject negation in Figure 2 and the octagon for knowledge representation in Figure 4 exhibit the same configuration of Aristotelian relations among their eight vertices.[20] In order to make this intuition more precise, we introduce the notion of an Aristotelian isomorphism (Demey and Smessaert, 2018b).

Definition 6. Consider two Boolean algebras $\mathbb{B}$ and $\mathbb{B}'$, and two fragments $\mathcal{F} \subseteq \mathbb{B}$ and $\mathcal{F}' \subseteq \mathbb{B}'$. An *Aristotelian isomorphism* is a bijection $\gamma\colon \mathcal{F} \to \mathcal{F}'$ such that for all relations $R \in \mathcal{AG}_{\mathbb{B}}$ and for all $x, y \in \mathcal{F}$: $R_{\mathbb{B}}(x, y)$ iff $R_{\mathbb{B}'}(\gamma(x), \gamma(y))$.

[19] Ciucci et al. (2016, p. 355) also assume that the transposed relations R^t and $\overline{R^t} = \overline{R}^t$ are serial, but those additional assumptions are only needed to construct yet another Keynes-Johnson octagon, starting from a non-trivial subset $T \subseteq X$ (i.e., $\emptyset \neq T \neq X$).

[20] This is by no means surprising, given that the characterizations of the X-subsets in Definition 5 take on the form of categorical statements with subject negation. However, later in this paper we will see that the same situation can also arise without any underlying syntactic similarity.

Recall that $\mathcal{F}_{sn}$ and $\mathcal{F}_{kr}$ are the sets of propositions/sets appearing in Figures 2 and Figure 4, respectively (cf. Definitions 2 and 5). It is now easy to check that the function $\gamma\colon \mathcal{F}_{sn} \to \mathcal{F}_{kr}$ defined below is indeed an Aristotelian isomorphism. For example, note that $\forall x(Sx \to Px)$ and $\forall x(Sx \to \neg Px)$ are EFOL-contraries, while $\gamma(\forall x(Sx \to Px))$ and $\gamma(\forall x(Sx \to \neg Px))$, i.e., $\overline{\overline{R}(S)}$ and $\overline{R(S)}$, are $\wp(X)$-contraries.

$$
\begin{array}{ccc}
\varphi \in \mathcal{F}_{sn} & \mapsto & \gamma(\varphi) \in \mathcal{F}_{kr} \\
\forall x(Sx \to Px) & \mapsto & \overline{\overline{R}(S)} \\
\exists x(Sx \wedge Px) & \mapsto & R(S) \\
\forall x(Sx \to \neg Px) & \mapsto & \overline{R(S)} \\
\exists x(Sx \wedge \neg Px) & \mapsto & \overline{R}(S) \\
\forall x(\neg Sx \to Px) & \mapsto & \overline{\overline{R}(\overline{S})} \\
\exists x(\neg Sx \wedge Px) & \mapsto & R(\overline{S}) \\
\forall x(\neg Sx \to \neg Px) & \mapsto & \overline{R(\overline{S})} \\
\exists x(\neg Sx \wedge \neg Px) & \mapsto & \overline{R}(\overline{S})
\end{array}
$$

The octagon for subject negation in Figure 2 and the octagon for knowledge representation in Figure 4 are thus Aristotelian isomorphic to each other. Using more classification-oriented terminology, we say that these two diagrams belong to the same *Aristotelian family*,[21] which will be called the 'family of Keynes-Johnson (KJ) octagons', since Keynes (1894) and Johnson (1921) were the first authors to study a member of this family. It should be emphasized that apart from Keynes and Johnson's octagon for subject negation, this Aristotelian family also contains (infinitely) many diagrams that were unknown to these historical authors. In particular, it makes perfect sense to say that the octagon for knowledge representation in Figure 4 belongs to the family of Keynes-Johnson octagons, without thereby implying that Keynes or Johnson themselves knew about knowledge representation, formal concept analysis, rough set theory, etc.[22]

Just like in the previous section, we now determine the bitstring semantics for the Keynes-Johnson octagon for knowledge representation. Recall that $\mathcal{F}_{kr} \subseteq \wp(X)$ is the set of eight X-subsets that appear in this octagon (cf. Definition 5). In order to determine the partition of the Boolean algebra $\wp(X)$ that is induced by $\mathcal{F}_{kr}$, we have to consider all intersections of (possibly X-complemented) sets from $\mathcal{F}_{kr}$. By systematically going through all intersections of this form and discarding those that are empty, we find the following partition:

[21] Formally, an Aristotelian family is a maximal class of Aristotelian isomorphic diagrams, i.e., a class $\mathcal{C}$ such that (i) any two diagrams belonging to $\mathcal{C}$ are Aristotelian isomorphic to each other, and (ii) if D belongs to $\mathcal{C}$, and D is Aristotelian isomorphic to D', then D' also belongs to $\mathcal{C}$ (Demey, 2018).

[22] Consider the following analogy. In abstract algebra, the class of Abelian groups is named after the mathematician Niels Abel (1802 – 1829), and contains (infinitely) many groups. Some members of this class were studied by the historical author Niels Abel, but it also contains (infinitely) many groups that were unknown to Abel in the early 19th century. Also cf. Demey (2019a, Footnote 14).

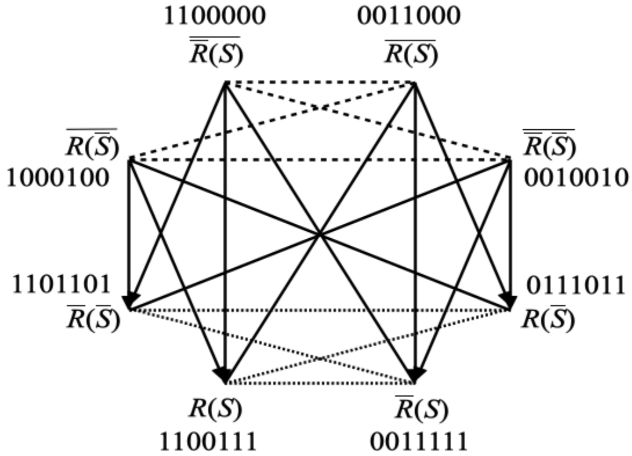

Figure 5: Bitstring semantics β_{kr} for the octagon of opposition for $(\mathcal{F}_{kr}, \wp(X))$.

$$
\begin{array}{rclrcl}
\Pi_{\wp(X)}(\mathcal{F}_{kr}) & = & \{ & A_1 & := & \overline{\overline{R}(S)} \cap \overline{R(\overline{S})}, \\
& & & A_2 & := & \overline{\overline{R}(S)} \cap R(\overline{S}), \\
& & & A_3 & := & \overline{R(S)} \cap \overline{\overline{R}(\overline{S})}, \\
& & & A_4 & := & \overline{R(S)} \cap \overline{R}(\overline{S}), \\
& & & A_5 & := & \overline{R}(S) \cap \overline{R(\overline{S})}, \\
& & & A_6 & := & R(S) \cap \overline{\overline{R}(\overline{S})}, \\
& & & A_7 & := & R(S) \cap \overline{R}(S) \cap R(\overline{S}) \cap \overline{R}(\overline{S}) \quad \}.
\end{array}
$$

For ease of notation, we will write the bitstring semantics $\beta^{\mathcal{F}_{kr}}_{\wp(X)}$ that corresponds to this partition simply as β_{kr}. Since $|\Pi_{\wp(X)}(\mathcal{F}_{kr})| = 7$, the Keynes-Johnson octagon for knowledge representation can also be represented by means of bitstrings of length 7. For example, since $\overline{\overline{R}(S)} = A_1 \cup A_2$, it can be represented as the bitstring 1100000, i.e., $\beta_{kr}(\overline{\overline{R}(S)}) = 1100000$. The bitstrings of all eight X-subsets in $\mathcal{F}_{kr}$ are shown in Figure 5. Finally, since $|\Pi_{\wp(X)}(\mathcal{F}_{kr})| = 7$, it follows that the Boolean closure of the Keynes-Johnson octagon for knowledge representation is isomorphic to $\{0,1\}^7$, and thus contains $2^7 = 128$ elements. This answers a question that was left open by Ciucci et al. (2016), who restricted themselves to noting that "the Boolean closure of the cube [i.e., octagon] is more complicated to compute than the Boolean closure of the square" (p. 361).

We have already seen that the octagons for subject negation ($\mathcal{F}_{sn}$) and for knowledge representation ($\mathcal{F}_{kr}$) are Aristotelian isomorphic: both are Keynes-Johnson octagons. In particular, the function $\gamma \colon \mathcal{F}_{sn} \to \mathcal{F}_{kr}$ preserves and reflects all Aristotelian relations. However, since these two octagons have the 'same' Boolean closure — viz., $\{0,1\}^7$, up to isomorphism —, we can prove something stronger: the function γ not only preserves and reflects all *Aristotelian* relations, but also all *Boolean* structure as well. More precisely: γ is not only an Aristotelian isomorphism, but also a Boolean isomorphism (Demey and Smessaert, 2018b).

Definition 7. Consider two Boolean algebras $\mathbb{B}$ and $\mathbb{B}'$, and two fragments $\mathcal{F} \subseteq \mathbb{B}$ and $\mathcal{F}' \subseteq \mathbb{B}'$. A *Boolean isomorphism* is a bijection $\gamma \colon \mathcal{F} \to \mathcal{F}'$ that can be extended to a Boolean algebra isomorphism $f \colon \mathbb{B}(\mathcal{F}) \to \mathbb{B}'(\mathcal{F}')$, i.e., such that $\gamma = f \restriction \mathcal{F}$.

$$
\begin{tikzcd}
\mathcal{F}_{sn} \arrow[rr, "\gamma"] \arrow[d, hook] & & \mathcal{F}_{kr} \arrow[d, hook] \\
\mathbb{B}_{\mathsf{EFOL}}(\mathcal{F}_{sn}) \arrow[rr, "f"] \arrow[dr, "\beta_{sn}"'] & & \mathbb{B}_{\wp(X)}(\mathcal{F}_{kr}) \arrow[dl, "\beta_{kr}"] \\
& \{0,1\}^7 &
\end{tikzcd}
$$

Figure 6: Commutative diagram for γ, f, β_{sn} and β_{kr}.

A Boolean isomorphism $\gamma\colon \mathcal{F} \to \mathcal{F}'$ preserves all Boolean operations, to the extent that they are defined inside of $\mathcal{F}$, i.e., for all $\varphi, \psi \in \mathcal{F}$, if $\neg\varphi \in \mathcal{F}$ then $\gamma(\neg\varphi) = \neg\gamma(\varphi)$, and if $\varphi \wedge \psi \in \mathcal{F}$, then $\gamma(\varphi \wedge \psi) = \gamma(\varphi) \wedge \gamma(\psi)$. The notion of Boolean isomorphism is strictly stronger than that of Aristotelian isomorphism: it can be shown that every Boolean isomorphism is an Aristotelian isomorphism, but not vice versa. We will encounter an example of an Aristotelian isomorphism that is not a Boolean isomorphism later in this paper.

To see that $\gamma\colon \mathcal{F}_{sn} \to \mathcal{F}_{kr}$ is a Boolean isomorphism, consider the function $f := \beta_{kr}^{-1} \circ \beta_{sn}\colon \mathbb{B}_{\mathsf{EFOL}}(\mathcal{F}_{sn}) \to \mathbb{B}_{\wp(X)}(\mathcal{F}_{kr})$. Since the bitstring semantics β_{sn} and β_{kr} are Boolean algebra isomorphisms, and this notion is closed under taking inverses and composition, the function f is a Boolean algebra isomorphism as well. Furthermore, it is easy to see that $\gamma = f \upharpoonright \mathcal{F}_{sn}$. Consider, for example, the formula $\forall x(Sx \to Px)$ from $\mathcal{F}_{sn}$, and note that $f(\forall x(Sx \to Px)) = \beta_{kr}^{-1}(\beta_{sn}(\forall x(Sx \to Px))) = \beta_{kr}^{-1}(1100000) = \overline{\overline{R}(S)} = \gamma(\forall x(Sx \to Px))$. Informally, we first use β_{sn} to 'encode' an element of $\mathcal{F}_{sn}$ as a bitstring of length 7, and consequently use β_{kr}^{-1} to 'decode' this bitstring into an element of $\mathcal{F}_{kr}$.[23] All of this is captured by the commutative diagram in Figure 6: (i) since the upper rectangle commutes, it holds that $\gamma = f \upharpoonright \mathcal{F}_{sn}$, and (ii) since the lower triangle commutes, it holds that $\beta_{sn} = \beta_{kr} \circ f$, i.e., $\beta_{kr}^{-1} \circ \beta_{sn} = f$.

4 An Octagon for Deontic Logic

In this section we will discuss one more Keynes-Johnson octagon. As far as we know, this octagon has not appeared in the literature thus far, but it arises quite naturally in the context of deontic logic. In particular, it can be viewed as a preliminary attempt to combine the literature on *normative positions* (Kanger, 1971; Lindahl, 1977; Makinson, 1986; Sergot, 2001, 2013) with that on *supererogation* (Chisholm, 1963; McNamara, 1996a,b; Joerden, 1998, 2012). We work with the deontic O-, P- and F-operators, which express that something is resp. obligatory, permitted and forbidden, and we make the standard assumptions that O and P are each other's duals (i.e., $O\varphi \equiv \neg P \neg\varphi$), that $F\varphi \equiv O\neg\varphi$, and that O is governed by the modal logic KD (i.e.,

[23] If desired, the Boolean algebra isomorphism f can also be obtained without appealing to bitstrings. Note that $\mathbb{B}_{\mathsf{EFOL}}(\mathcal{F}_{sn})$ and $\mathbb{B}_{\wp(X)}(\mathcal{F}_{kr})$ are both complete atomic Boolean algebras, which have $\Pi_{\mathsf{EFOL}}(\mathcal{F}_{sn})$ and $\Pi_{\wp(X)}(\mathcal{F}_{kr})$ as their sets of atoms, respectively. Since these two Boolean algebras have the same number of atoms (viz., 7), they are isomorphic (Givant and Halmos, 2009, pp. 121–122).

$O\varphi$ entails $P\varphi$ but does not entail φ). In this framework, we will consider the formulas of the form $(\neg)p \wedge P(\neg)p$ and $(\neg)p \vee O(\neg)p$.

Definition 8. The eight propositions that appear in the Keynes-Johnson octagon for deontic logic:

1.	$p \wedge Pp$		
2.	$p \wedge P\neg p$		
3.	$\neg p \wedge Pp$		
4.	$\neg p \wedge P\neg p$		
5.	$\neg p \vee O\neg p$	i.e.,	$p \to Fp$
6.	$\neg p \vee Op$	i.e.,	$p \to Op$
7.	$p \vee O\neg p$	i.e.,	$\neg p \to Fp$
8.	$p \vee Op$	i.e.,	$\neg p \to Op$

The set consisting of these eight propositions will be denoted $\mathcal{F}_{dl}$.

These propositions describe certain situations that naturally occur in everyday life. For example, suppose that p states that John goes to the party. Propositions 1 and 4 describe perfectly mundane situations; for example, proposition 1 states that John goes to the party and it is indeed permitted that he goes. By contrast, proposition 2 and 3 require a more nuanced perspective. For example, proposition 2 states that John goes to the party even though it is permitted that he does not go (John's going to the party might thus amount to an act of supererogation). Furthermore, propositions 6 and 7 can be seen as describing a conscientious person, who actively tries to meet her deontic requirements. For example, proposition 6 states that John goes to the party only if he is obliged to do so (one can easily imagine that John would rather not go to the party, but only does so in order to meet his obligations). By contrast, proposition 5 and 8 can be seen as describing a rebellious or contrarian person, who actively tries to violate her deontic requirements. For example, proposition 5 states that John goes to the party only if he is forbidden from doing so (one can imagine that John has no intrinsic interest in going to the party, but only does so if he is forbidden from actually going).

As was already mentioned above, we view the eight propositions of Definition 8 as coming from the logical system KD. By plugging the Lindenbaum-Tarski algebra $\mathbb{B}(\mathsf{KD})$ into Definition 1, we can thus characterize the Aristotelian relations that obtain between these eight propositions, as shown in the Keynes-Johnson octagon in Figure 7. For example, in order to see that $p \wedge Pp$ and $\neg p \wedge Pp$ are KD-contrary, it suffices to check the following two conditions:

- $\models_{\mathsf{KD}} \neg((p \wedge Pp) \wedge (\neg p \wedge Pp))$

  After all, there exist no KD-model $\mathbb{M}$ and possible world w such that $\mathbb{M}, w \models (p \wedge Pp) \wedge (\neg p \wedge Pp)$, since that would entail $\mathbb{M}, w \models p \wedge \neg p$. This means that $\models_{\mathsf{KD}} \neg((p \wedge Pp) \wedge (\neg p \wedge Pp))$.

- $\not\models_{\mathsf{KD}} (p \wedge Pp) \vee (\neg p \wedge Pp)$

  After all, there exist a KD-model $\mathbb{M}$ and possible world w such that $\mathbb{M}, w \not\models Pp$, and thus also $\mathbb{M}, w \not\models (p \wedge Pp) \vee (\neg p \wedge Pp)$. This means that $\not\models_{\mathsf{KD}} (p \wedge Pp) \vee (\neg p \wedge Pp)$.

Upon visual inspection, it is again immediately clear that the octagon for subject negation in Figure 2 and the octagon for deontic logic in Figure 7 exhibit the same

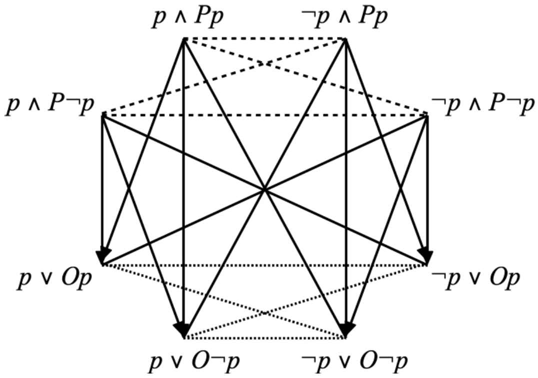

Figure 7: Octagon of opposition for $(\mathcal{F}_{dl}, \mathsf{KD})$.

configuration of Aristotelian relations among their eight vertices.[24] More formally, it is easy to check that the function $\delta\colon \mathcal{F}_{sn} \to \mathcal{F}_{dl}$ defined below is indeed an Aristotelian isomorphism (cf. Definition 6). For example, note that $\forall x(Sx \to Px)$ and $\forall x(Sx \to \neg Px)$ are EFOL-contraries, while $\delta(\forall x(Sx \to Px))$ and $\delta(\forall x(Sx \to \neg Px))$, i.e., $p \wedge Pp$ and $\neg p \wedge Pp$, are KD-contraries. Using more classification-oriented terminology, we say that the octagon for deontic logic in Figure 7 also belongs to the Aristotelian family of Keynes-Johnson octagons.

$$
\begin{array}{ccc}
\varphi \in \mathcal{F}_{sn} & \mapsto & \delta(\varphi) \in \mathcal{F}_{dl} \\
\forall x(Sx \to Px) & \mapsto & p \wedge Pp \\
\exists x(Sx \wedge Px) & \mapsto & p \vee O\neg p \\
\forall x(Sx \to \neg Px) & \mapsto & \neg p \wedge Pp \\
\exists x(Sx \wedge \neg Px) & \mapsto & \neg p \vee O\neg p \\
\forall x(\neg Sx \to Px) & \mapsto & \neg p \wedge P\neg p \\
\exists x(\neg Sx \wedge Px) & \mapsto & \neg p \vee Op \\
\forall x(\neg Sx \to \neg Px) & \mapsto & p \wedge P\neg p \\
\exists x(\neg Sx \wedge \neg Px) & \mapsto & p \vee Op
\end{array}
$$

Just like in the previous two sections, we now determine the bitstring semantics for the Keynes-Johnson octagon for deontic logic. First of all, we compute the partition of the Boolean algebra $\mathbb{B}(\mathsf{KD})$ that is induced by $\mathcal{F}_{dl}$:

$$
\begin{array}{lclcll}
\Pi_{\mathsf{KD}}(\mathcal{F}_{dl}) & = & \{ & \alpha'_1 & := & p \wedge Pp \wedge P\neg p, \\
 & & & \alpha'_2 & := & p \wedge Op, \\
 & & & \alpha'_3 & := & \neg p \wedge Pp \wedge P\neg p, \\
 & & & \alpha'_4 & := & \neg p \wedge Op, \\
 & & & \alpha'_5 & := & p \wedge O\neg p, \\
 & & & \alpha'_6 & := & \neg p \wedge O\neg p \qquad \}.
\end{array}
$$

For ease of notation, we will write the bitstring semantics $\beta^{\mathcal{F}_{dl}}_{\mathsf{KD}}$ that corresponds to this partition simply as β_{dl}. Since $|\Pi_{\mathsf{KD}}(\mathcal{F}_{dl})| = 6$, the Keynes-Johnson octagon for deontic logic can be represented by means of bitstrings of length 6. For example, since $p \wedge Pp$ is KD-equivalent to $\alpha'_1 \vee \alpha'_2$, it can be represented as the bitstring 110000,

[24]This Aristotelian isomorphism obtains even though there is no clear-cut underlying syntactic similarity between the propositions that appear in both octagons; recall Footnote 20.

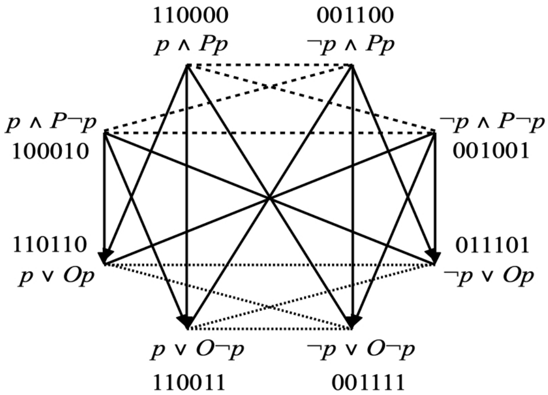

Figure 8: Bitstring semantics β_{dl} for the octagon of opposition for $(\mathcal{F}_{dl}, \mathsf{KD})$.

i.e., $\beta_{dl}(p \wedge Pp) = 110000$. The bitstrings of all eight propositions in $\mathcal{F}_{dl}$ are shown in Figure 8. Finally, since $|\Pi_{\mathsf{KD}}(\mathcal{F}_{dl})| = 6$, it follows that the Boolean closure of the Keynes-Johnson octagon for deontic logic is isomorphic to $\{0,1\}^6$, and thus contains $2^6 = 64$ elements.

We have already seen that the function $\delta\colon \mathcal{F}_{sn} \to \mathcal{F}_{dl}$ is an Aristotelian isomorphism: it preserves and reflects all Aristotelian relations (Definition 6). However, δ is clearly not a Boolean isomorphism: it does not preserve all Boolean operations (Definition 7). For example, note that the disjunction of the four uppermost formulas in Figure 2, i.e., $\forall x(Sx \to Px)$, $\forall x(Sx \to \neg Px)$, $\forall x(\neg Sx \to Px)$ and $\forall x(\neg Sx \to \neg Px)$, is not an EFOL-tautology,[25] whereas the disjunction of their δ-images, i.e., $p \wedge Pp$, $\neg p \wedge Pp$, $\neg p \wedge P\neg p$ and $p \wedge P\neg p$ in Figure 7, is a KD-tautology.[26]

It is also interesting to examine how δ interacts with the partitions $\Pi_{\mathsf{EFOL}}(\mathcal{F}_{sn})$ and $\Pi_{\mathsf{KD}}(\mathcal{F}_{dl})$. For example, recall that $\alpha_1 \in \Pi_{\mathsf{EFOL}}(\mathcal{F}_{sn})$ is the conjunction $\forall x(Sx \to Px) \wedge \forall x(\neg Sx \to \neg Px)$, so if we apply δ to both conjuncts,[27] we obtain $\delta(\forall x(Sx \to Px)) \wedge \delta(\forall x(\neg Sx \to \neg Px)) = (p \wedge Pp) \wedge (p \wedge P\neg p) \equiv_{\mathsf{KD}} p \wedge Pp \wedge P\neg p = \alpha'_1 \in \Pi_{\mathsf{KD}}(\mathcal{F}_{dl})$. This correspondence between $\alpha_i \in \Pi_{\mathsf{EFOL}}(\mathcal{F}_{sn})$ and $\alpha'_i \in \Pi_{\mathsf{KD}}(\mathcal{F}_{dl})$ holds for $1 \leq i \leq 6$. If we try to determine the counterpart of α_7 in the same fashion, we find $\delta(\exists x(Sx \wedge Px)) \wedge \delta(\exists x(Sx \wedge \neg Px)) \wedge \delta(\exists x(\neg Sx \wedge Px)) \wedge \delta(\exists x(\neg Sx \wedge \neg Px)) = (p \vee O\neg p) \wedge (\neg p \vee O\neg p) \wedge (\neg p \vee Op) \wedge (p \vee Op)$, which turns out to be KD-inconsistent; in other words, $\alpha_7 \in \Pi_{\mathsf{EFOL}}(\mathcal{F}_{sn})$ does not have a counterpart in $\Pi_{\mathsf{KD}}(\mathcal{F}_{dl})$. This means that the bitstring representations of $\mathcal{F}_{dl}$ can be viewed as the result of systematically deleting the seventh bit position in the bitstring representations of $\mathcal{F}_{sn}$ (compare Figures 3 and 8). This process of deleting one bit position does not have any effect on the octagons' Aristotelian relations (they are Aristotelian isomorphic!), but as we have seen above, it does have a significant effect on their Boolean structure.

To make this formally precise, let $d_7\colon \{0,1\}^7 \to \{0,1\}^6$ be the function that deletes a bitstring's seventh bit position, and consider the function $g := \beta_{dl}^{-1} \circ d_7 \circ \beta_{sn}\colon \mathbb{B}_{\mathsf{EFOL}}(\mathcal{F}_{sn}) \to \mathbb{B}_{\mathsf{KD}}(\mathcal{F}_{dl})$. It is easy to see that $\delta = g \restriction \mathcal{F}_{sn}$.[28] Consider, for

[25]In terms of bitstrings: $\beta_{sn}(\forall x(Sx \to Px)) \vee \beta_{sn}(\forall x(Sx \to \neg Px)) \vee \beta_{sn}(\forall x(\neg Sx \to Px)) \vee \beta_{sn}(\forall x(\neg Sx \to \neg Px)) = 1100000 \vee 0011000 \vee 0010010 \vee 1000100 = 1111110 \neq 1111111$.

[26]In terms of bitstrings: $\beta_{dl}(p \wedge Pp) \vee \beta_{dl}(\neg p \wedge Pp) \vee \beta_{dl}(\neg p \wedge P\neg p) \vee \beta_{dl}(p \wedge P\neg p) = 110000 \vee 001100 \vee 001001 \vee 100010 = 111111$.

[27]Note that we can only apply δ to the conjuncts separately, and not to the entire conjunction, since that conjunction does not belong to $\mathcal{F}_{sn}$ and thus falls outside the domain of δ.

[28]Note that this does *not* mean that δ is a Boolean isomorphism, since g itself is not a Boolean algebra

$$\begin{tikzcd}
\mathcal{F}_{sn} \arrow[r, "\delta"] \arrow[d, hook] & \mathcal{F}_{dl} \arrow[d, hook] \\
\mathbb{B}_{\mathsf{EFOL}}(\mathcal{F}_{sn}) \arrow[r, "g"] \arrow[d, "\beta_{sn}"'] & \mathbb{B}_{\mathsf{KD}}(\mathcal{F}_{dl}) \arrow[d, "\beta_{dl}"] \\
\{0,1\}^7 \arrow[r, "d_7"] & \{0,1\}^6
\end{tikzcd}$$

Figure 9: Commutative diagram for δ, g, d_7, β_{sn} and β_{dl}.

example, the formula $\forall x(Sx \to Px)$ from $\mathcal{F}_{sn}$, and note that $g(\forall x(Sx \to Px)) = \beta_{dl}^{-1}(d_7(\beta_{sn}(\forall x(Sx \to Px)))) = \beta_{dl}^{-1}(d_7(1100000)) = \beta_{dl}^{-1}(110000) = p \wedge Pp = \delta(\forall x(Sx \to Px))$. Informally, we first use β_{sn} to 'encode' an element of $\mathcal{F}_{sn}$ as a bitstring of length 7, then use d_7 to delete the 7th bit position of that bitstring, and finally use β_{dl}^{-1} to 'decode' the resulting bitstring of length 6 into an element of $\mathcal{F}_{dl}$. All of this is captured by the commutative diagram in Figure 9: (i) since the upper rectangle commutes, it holds that $\delta = g \upharpoonright \mathcal{F}_{sn}$, and (ii) since the lower rectangle commutes, it holds that $d_7 \circ \beta_{sn} = \beta_{dl} \circ g$, i.e., $\beta_{dl}^{-1} \circ d_7 \circ \beta_{sn} = g$.

We have already seen that $\delta \colon \mathcal{F}_{sn} \to \mathcal{F}_{dl}$ is not a Boolean isomorphism. Furthermore, there does not exist *any* Boolean isomorphism between $\mathcal{F}_{sn}$ and $\mathcal{F}_{dl}$. After all, if there did exist a Boolean isomorphism between these two fragments, then by Definition 7 there would also exist a Boolean algebra isomorphism between their Boolean closures; however, we have already seen that that these two Boolean closures are isomorphic to $\{0,1\}^7$ and $\{0,1\}^6$, respectively, and are thus not isomorphic to each other. To summarize: the octagon for subject negation ($\mathcal{F}_{sn}$) and the octagon for deontic logic ($\mathcal{F}_{dl}$) are *Aristotelian isomorphic*, but they are not *Boolean isomorphic*.[29]

Given its importance, it is worthwhile to repeat this conclusion once more, but now using more classification-oriented terminology: the octagon for subject negation ($\mathcal{F}_{sn}$) and the octagon for deontic logic ($\mathcal{F}_{dl}$) belong to the *same Aristotelian family* (viz., the family of Keynes-Johnson octagons), but they belong to *different Boolean subfamilies* of this Aristotelian family (viz., the subfamily of KJ octagons that are representable by bitstrings of length 7 and the subfamily of KJ octagons that are representable by bitstrings of length 6, respectively).

isomorphism (d_7 is not a Boolean algebra isomorphism either).

[29] We are now in an ideal position to briefly revisit Reichenbach (1952)'s diagram in REFOL (also cf. Footnote 11). In particular, Dubois et al. (2020) claim that Reichenbach's diagram is isomorphic to that of Moretti (2009) (rather than to those of Keynes (1894) and Johnson (1921), as they had previously argued). This claim is correct, but it would be good to be more precise in the formulation: Reichenbach's and Moretti's diagrams are indeed *Aristotelian* isomorphic, but not *Boolean* isomorphic to each other. In particular, Moretti's diagram can be represented by bitstrings of length 4, whereas Reichenbach's diagram requires bitstrings of length 5 (Demey, 2020b, p. 189).

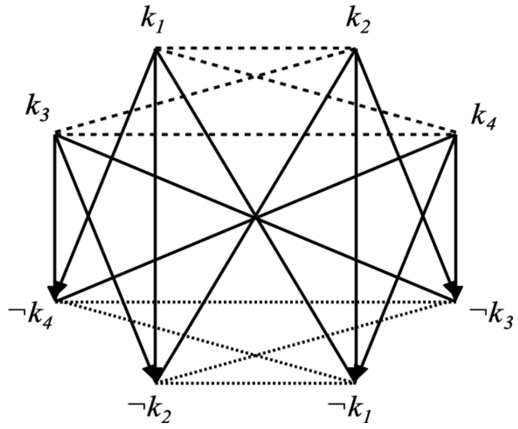

Figure 10: Generic description of the Aristotelian family of Keynes-Johnson octagons.

5 Broader Theoretical Perspective

In the previous three sections we have come across Keynes-Johnson octagons with Boolean closure isomorphic to $\{0,1\}^7$ (viz., $\mathcal{F}_{sn}$ and $\mathcal{F}_{kr}$) and with Boolean closure isomorphic to $\{0,1\}^6$ (viz., $\mathcal{F}_{dl}$). This shows that the Aristotelian family of Keynes-Johnson octagons has *at least two* distinct Boolean subfamilies. It is thus natural to ask whether there are any other Boolean subfamilies besides these two: do there exist Keynes-Johnson octagons with a Boolean closure that is isomorphic to $\{0,1\}^n$, for some $n \notin \{6,7\}$? In this section we will use the general results of Demey (2018) to show that this is not the case. In other words, the Aristotelian family of Keynes-Johnson octagons has *precisely two* Boolean subfamilies.

We start by drawing a distinction between an Aristotelian family and a 'generic description' of that family. The former is an infinite collection (a proper class, even) of concrete Aristotelian diagrams coming from all kinds of Boolean algebras; the latter is a more abstract description of that family, which does not refer to any specific Boolean algebra, but just specifies a configuration of Aristotelian relations holding between elements. For example, the octagons in Figures 2, 4 and 7 are three concrete members of the Aristotelian family of Keynes-Johnson octagons, while the generic description of this family is shown in Figure 10. The set of elements that appear in the generic description of an Aristotelian family $\mathcal{A}$ is called $\mathcal{F}_{\mathcal{A}}$; for example, in Figure 10 we find that $\mathcal{F}_{KJ} = \{k_1, k_2, k_3, k_4, \neg k_1, \neg k_2, \neg k_3, \neg k_4\}$. We emphasize, once more, that these k_i do not come from any specific Boolean algebra, but rather function as abstract 'placeholders' to specify a configuration of Aristotelian relations.

We now attempt to compute the partition that is induced by $\mathcal{F}_{KJ}$. We cannot straightforwardly proceed as in the previous sections, since Definition 3 refers to a specific Boolean algebra $\mathbb{B}$, viz., in its requirement that conjunctions belonging to the partition should be $\mathbb{B}$-consistent (i.e., not equal to $\perp_{\mathbb{B}}$). However, given an Aristotelian family $\mathcal{A}$, we can 'approximate' this notion of $\mathbb{B}$-consistency without having to refer to any specific Boolean algebra $\mathbb{B}$.

Definition 9. Let $\mathcal{A}$ be an Aristotelian family, and let $\mathcal{F}_{\mathcal{A}} = \{x_1, \ldots, x_m\}$ be the set of elements that appear in the generic description of $\mathcal{A}$. A meet of $\mathcal{F}_{\mathcal{A}}$-elements is said to be $\mathcal{A}$-*consistent* iff it does not contain any pair of elements that are contradictory or contrary according to the generic description of $\mathcal{A}$.

The notion of $\mathcal{A}$-consistency is an '(upper) approximation' of $\mathbb{B}$-consistency, in the following sense. One can easily show that $\mathbb{B}$-consistency entails $\mathcal{A}$-consistency, but not vice versa. After all, it might happen that a meet of $\mathcal{F}_\mathcal{A}$-elements is $\mathcal{A}$-consistent while still being $\mathbb{B}$-inconsistent — namely, when the inconsistency cannot be 'pinpointed' to just *two* elements in the meet (this is a direct counterpart of the strictly *binary* nature of the Aristotelian relations). Having defined the notion of $\mathcal{A}$-consistency, we can now proceed to define the partition that is induced by $\mathcal{F}_\mathcal{A}$.

Definition 10. Let $\mathcal{F}_\mathcal{A}$ be as in Definition 9. The *partition induced by* $\mathcal{F}_\mathcal{A}$ is defined as $\Pi(\mathcal{F}_\mathcal{A}) := \{a \mid a = \pm x_1 \wedge \cdots \wedge \pm x_m$, and a is $\mathcal{A}$-consistent$\}$.

For example, given $\mathcal{F}_{KJ}$ as in Figure 10, we see that $k_1 \wedge \neg k_2 \wedge k_3 \wedge k_4$ is *KJ*-inconsistent, since the first and last element of this meet (i.e., k_1 and k_4) are contrary according to the generic description in Figure 10. By contrast, $k_1 \wedge \neg k_2 \wedge k_3 \wedge \neg k_4$ is *KJ*-consistent, since it does not contain any two elements that are contradictory or contrary according to the generic description in Figure 10. Furthermore, since this generic description specificies that k_1 is contrary to k_2 and k_4 (and thus in subalternation to $\neg k_2$ and $\neg k_4$), the meet $k_1 \wedge \neg k_2 \wedge k_3 \wedge \neg k_4$ can be simplified to $k_1 \wedge k_3$. By systematically going through all meets of this form, simplifying whenever possible, and discarding the *KJ*-inconsistent meets, we find the partition induced by the generic description of the family of Keynes-Johnson octagons:

$$\begin{array}{rclrcll}
\Pi(\mathcal{F}_{KJ}) & = & \{ & \varepsilon_1 & := & k_1 \wedge k_3, & \\
 & & & \varepsilon_2 & := & k_1 \wedge \neg k_3, & \\
 & & & \varepsilon_3 & := & k_2 \wedge k_4, & \\
 & & & \varepsilon_4 & := & k_2 \wedge \neg k_4, & \\
 & & & \varepsilon_5 & := & \neg k_1 \wedge k_3, & \\
 & & & \varepsilon_6 & := & \neg k_2 \wedge k_4, & \\
 & & & \varepsilon_7 & := & \neg k_1 \wedge \neg k_2 \wedge \neg k_3 \wedge \neg k_4 & \}
\end{array}$$

Let's consider ε_1 in some more detail. First of all, note that ε_1 is *KJ*-consistent, by the construction of $\Pi(\mathcal{F}_{KJ})$. Furthermore, ε_1 is also guaranteed to be $\mathbb{B}$-consistent (for any Boolean algebra $\mathbb{B}$).[30] After all, suppose that there exists some Boolean algebra $\mathbb{B}$ such that ε_1 is $\mathbb{B}$-inconsistent. Since ε_1 is the meet of only two $\mathcal{F}_{KJ}$-elements (after simplification), viz., k_1 and k_3 (which are themselves supposed to be $\mathbb{B}$-consistent), this would mean that those elements k_1 and k_3 are either $\mathbb{B}$-contradictory or $\mathbb{B}$-contrary. But this violates the *KJ*-consistency of ε_1. To summarize: since ε_1 is the meet of *at most two* $\mathcal{F}_{KJ}$-elements, its *KJ*-consistency suffices to guarantee its $\mathbb{B}$-consistency (for any $\mathbb{B}$). Since ε_2 – ε_6 are also the meets of only two $\mathcal{F}_{KJ}$-elements, exactly the same remarks apply to them as well.

This situation stands in sharp contrast with that of ε_7. Again, ε_7 is *KJ*-consistent, by the construction of $\Pi(\mathcal{F}_{KJ})$. However, since ε_7 is the meet of *more than two* $\mathcal{F}_{KJ}$-elements (viz., $\neg k_1$, $\neg k_2$, $\neg k_3$ and $\neg k_4$), it is not guaranteed to be $\mathbb{B}$-consistent (for every Boolean algebra $\mathbb{B}$). In other words, there exist Boolean algebras in which ε_7 is

[30] Here are two concrete examples. First of all, with respect to the Boolean algebra $\mathbb{B}(\mathsf{EFOL})$ and the fragment $\mathcal{F}_{sn}$ (cf. Figure 2), the placeholders k_1 and k_3 get interpreted as $\forall x(Sx \rightarrow Px)$ and $\forall x(\neg Sx \rightarrow \neg Px)$, respectively, and hence, $\varepsilon_1 = k_1 \wedge k_3$ gets interpreted as $\forall x(Sx \rightarrow Px) \wedge \forall x(\neg Sx \rightarrow \neg Px)$, which is EFOL-consistent. Secondly, with respect to the Boolean algebra $\mathbb{B}(\mathsf{KD})$ and the fragment $\mathcal{F}_{dl}$ (cf. Figure 7), the placeholders k_1 and k_3 get interpreted as $p \wedge Pp$ and $p \wedge P\neg p$, respectively, and hence, $\varepsilon_1 = k_1 \wedge k_3$ gets interpreted as $(p \wedge Pp) \wedge (p \wedge P\neg p)$, which is KD-consistent.

consistent, but there also exist Boolean algebras in which ε_7 is inconsistent. Consider the following examples (and compare with Footnote 30):

- the Boolean algebra $\mathbb{B}(\mathsf{EFOL})$ and the fragment $\mathcal{F}_{sn}$ (cf. Figure 2)

  The placeholders k_1, k_2, k_3 and k_4 get interpreted as $\forall x(Sx \to Px)$, $\forall x(Sx \to \neg Px)$, $\forall x(\neg Sx \to \neg Px)$ and $\forall x(\neg Sx \to Px)$, respectively. Hence, $\varepsilon_7 = \neg k_1 \wedge \neg k_2 \wedge \neg k_3 \wedge \neg k_4$ gets interpreted as $\exists x(Sx \wedge \neg Px) \wedge \exists x(Sx \wedge Px) \wedge \exists x(\neg Sx \wedge Px) \wedge \exists x(\neg Sx \wedge \neg Px)$, which is EFOL-consistent.

- the Boolean algebra $\mathbb{B}(\mathsf{KD})$ and the fragment $\mathcal{F}_{dl}$ (cf. Figure 7)

  The placeholders k_1, k_2, k_3 and k_4 get interpreted as $p \wedge Pp$, $\neg p \wedge Pp$, $p \wedge P\neg p$ and $\neg p \wedge P\neg p$, respectively. Hence, $\varepsilon_7 = \neg k_1 \wedge \neg k_2 \wedge \neg k_3 \wedge \neg k_4$ gets interpreted as $(\neg p \vee \neg Pp) \wedge (p \vee \neg Pp) \wedge (\neg p \vee \neg P\neg p) \wedge (p \vee \neg P\neg p)$, which is KD-inconsistent. (We emphasize that this conjunction is still *KJ*-consistent, since it does not contain any pair of conjuncts that are contradictory or contrary according to the generic description of Keynes-Johnson octagons.)

We are now in a position to summarize the entire argument. The partition $\Pi(\mathcal{F}_{KJ})$ consists of 7 *KJ*-consistent conjunctions. The formulas $\varepsilon_1, \ldots, \varepsilon_6$ have at most two conjuncts, and hence they are guaranteed by the construction of $\Pi(\mathcal{F}_{KJ})$ to be $\mathbb{B}$-consistent as well (for any Boolean algebra $\mathbb{B}$). By contrast, ε_7 has more than two conjuncts, and hence it is not guaranteed to be $\mathbb{B}$-consistent. We thus have to make a case distinction. On the one hand, there are Boolean algebras $\mathbb{B}$ and KJ octagons $\mathcal{F} \subseteq \mathbb{B}$ such that ε_7 is $\mathbb{B}$-consistent, and hence $|\Pi_{\mathbb{B}}(\mathcal{F})| = 7$. On the other hand, there are Boolean algebras $\mathbb{B}$ and KJ octagons $\mathcal{F} \subseteq \mathbb{B}$ such that ε_7 is $\mathbb{B}$-inconsistent, and hence $|\Pi_{\mathbb{B}}(\mathcal{F})| = 6$.

Since this is an exhaustive case analysis, we can thus conclude that the Aristotelian family of Keynes-Johnson octagons has exactly two Boolean subfamilies: those which induce a partition of 7 elements and thus have a Boolean closure isomorphic to $\{0,1\}^7$, and those which induce a partition of 6 elements and thus have a Boolean closure isomorphic to $\{0,1\}^6$. The Keynes-Johnson octagons discussed in Sections 2 and 3 both belong to the former Boolean subfamily, while the Keynes-Johnson octagon discussed in Section 4 belongs to the latter.

6 Conclusion

In this paper, we have analyzed three concrete octagons of opposition, viz., an octagon for subject negation ($\mathcal{F}_{sn}$, Figure 2), an octagon for knowledge representation ($\mathcal{F}_{kr}$, Figure 4), and an octagon for deontic logic ($\mathcal{F}_{dl}$, Figure 7). We have shown that these three octagons are all Aristotelian isomorphic to each other: they all belong to the same Aristotelian family, viz., the family of Keynes-Johnson (KJ) octagons. Furthermore, we have also shown that the first two octagons are Boolean isomorphic to each other: they belong to the same Boolean subfamily, viz., the subfamily of KJ octagons that are representable by bitstrings of length 7, and thus have Boolean closures isomorphic to $\{0,1\}^7$. Finally, we have shown that the third octagon is not Boolean isomorphic to the first two: it belongs to a different Boolean subfamily, viz., the subfamily of KJ octagons that are representable by bitstrings of length 6, and thus have Boolean closures isomorphic to $\{0,1\}^6$.

The results obtained in this paper fit within the ongoing effort in logical geometry toward setting up a systematic classification of Aristotelian families and their Boolean subfamilies. For example, if we focus on just the *octagons* of opposition, one can show that there exist precisely 18 distinct Aristotelian families of octagons (Frijters and Demey, 2023). For each of these families, we should determine what its Boolean subfamilies are. Previously, Demey (2019a) has investigated the Aristotelian family of so-called *Buridan octagons*, which turns out to have three Boolean subfamilies (corresponding to bitstrings of lengths 4, 5 and 6). In this paper we have shown that the Aristotelian family of Keynes-Johnson octagons has precisely two Boolean subfamilies (corresponding to bitstrings of lengths 6 and 7). By gradually filling in the details of this systematic classification, we hope to obtain a unified perspective on Aristotelian diagrams and their various applications across history and across scientific disciplines (Demey, 2019c).

Acknowledgements. This research was funded by the research project 'BITSHARE: Bitstring Semantics for Human and Artificial Reasoning' (IDN-19-009, Internal Funds KU Leuven). A much earlier version of this paper was presented by the first author at the 6th World Congress on the Square of Opposition (1 – 5 November 2018, Crete, Greece). Next to the audience of that talk, we would like to thank Jean-Yves Beziau, Alex De Klerck, Atahan Erbas, Stef Frijters, Anna-Sophie Heinemann, Jan Heylen, Henri Prade, Margaux Smets and Joost Vennekens for their interesting remarks and suggestions. The first author holds a research professorship (BOFZAP) at KU Leuven.

References

J. L. Ackrill. *Aristotle – Categories and De Interpretatione*. Clarendon Press, 1963.

E. Alvarez and M. Correia. Conversion and opposition: Traditional and theoretical formulations. In J.-Y. Beziau and S. Gerogiorgakis, editors, *New Dimensions of the Square of Opposition*, pages 87–105. Philosophia Verlag, 2017.

E. Alvarez-Fontecilla. Canonical syllogistic moods in traditional Aristotelian logic. *Logica Universalis*, 20:517–531, 2016.

L. Amgoud and H. Prade. A formal concept view of formal argumentation. In L. C. van der Gaag, editor, *Symbolic and Quantiative Approaches to Reasoning with Uncertainty (ECSQARU 2013)*, LNCS 7958, pages 1–12. Springer, 2013.

A. Baker. Classical logical relations. *Notre Dame Journal of Formal Logic*, 18:164–168, 1977.

H. Ben-Yami. The quantified argument calculus. *Review of Symbolic Logic*, 7:120–146, 2014.

H. Ben-Yami. The quantified argument calculus and natural logic. *Dialectica*, forthcoming.

R. Carnap. *Meaning and Necessity. A Study in Semantics and Modal Logic*. University of Chicago Press, 1947.

R. M. Chisholm. Supererogation and offence: A conceptual scheme for ethics. *Ratio*, 5:1–14, 1963.

R. Christensen. The first square of opposition. *Phronesis*, 68:371–383, 2023.

I. Ciardelli, J. Groenendijk, and F. Roelofsen. *Inquisitive Semantics*. Oxford University Press, 2019.

D. Ciucci, D. Dubois, and H. Prade. Oppositions in rough set theory. In T. Li et al., editors, *Rough Sets and Knowledge Technology*, LNAI 7414, pages 504–513. Springer, 2012.

D. Ciucci, D. Dubois, and H. Prade. The structure of oppositions in rough set theory and formal concept analysis – Toward a new bridge between the two settings. In C. Beierle and C. Meghini, editors, *Foundations of Information and Knowledge Systems (FoIKS 2014)*, LNCS 8367, pages 154–173. Springer, 2014.

D. Ciucci, D. Dubois, and H. Prade. Structures of opposition in fuzzy rough sets. *Fundamenta Informaticae*, 142:1–19, 2015.

D. Ciucci, D. Dubois, and H. Prade. Structures of opposition induced by relations. The Boolean and the gradual cases. *Annals of Mathematics and Artificial Intelligence*, 76:351–373, 2016.

M. R. Cohen and E. Nagel. *An Introduction to Logic and Scientific Method*. Routledge and Kegan Paul, 1930.

M. Correia. The syllogistic theory of Boethius. *Ancient Philosophy*, 29:391–405, 2009.

M. Correia. The proto-exposition of Aristotelian categorical logic. In J.-Y. Beziau and G. Basti, editors, *The Square of Opposition: A Cornerstone of Thought*, pages 21–34. Springer, 2017.

H. B. Curry. A mathematical treatment of the rules of the syllogism. *Mind*, 45:209–216, 1936.

T. Czeżowski. *Odczyty filozoficzne [Philosophical Lectures]*. Towarzystwo Naukowe w Torunin, 1958.

A. De Klerck, L. Vignero, and L. Demey. Morphisms between Aristotelian diagrams. *Logica Universalis*, pages 1–35, 2023.

T. de Laguna. Opposition and the syllogism. *Journal of Philosophy, Psychology and Scientific Methods*, 9:393–400, 1912.

P. Dekker. Not *Only* Barbara. *Journal of Logic, Language and Information*, 24:95–129, 2015.

L. Demey. Interactively illustrating the context-sensitivity of Aristotelian diagrams. In H. Christiansen, I. Stojanovic, and G. Papadopoulos, editors, *Modeling and Using Context*, LNCS 9405, pages 331–345. Springer, 2015.

L. Demey. Computing the maximal Boolean complexity of families of Aristotelian diagrams. *Journal of Logic and Computation*, 28:1323–1339, 2018.

L. Demey. Boolean considerations on John Buridan's octagons of oppositions. *History and Philosophy of Logic*, 40:116–134, 2019a.

L. Demey. Metalogic, metalanguage and logical geometry. *Logique et Analyse*, 248: 453–478, 2019b.

L. Demey. The role of Aristotelian diagrams in scientific communication. In A. Benedek and K. Nyíri, editors, *Learning and Technology in Historical Perspective*, Perspectives on Visual Learning 2, pages 69–77. Hungarian Academy of Sciences, 2019c.

L. Demey. Mathematization and *Vergessenmachen* in the historiography of logic. *History of Humanities*, 5:51–74, 2020a.

L. Demey. From Euler diagrams in Schopenhauer to Aristotelian diagrams in logical geometry. In J. Lemanski, editor, *Language, Logic, and Mathematics in Schopenhauer*, pages 181–205. Springer, 2020b.

L. Demey. Aristotelian diagrams for semantic and syntactic consequence. *Synthese*, 198:187–207, 2021.

L. Demey and S. Frijters. Logic-sensitivity and bitstring semantics in the square of opposition. *Journal of Philosophical Logic*, 52:1703–1721, 2023.

L. Demey and H. Smessaert. The relationship between Aristotelian and Hasse diagrams. In T. Dwyer, H. Purchase, and A. Delaney, editors, *Diagrammatic Representation and Inference*, LNCS 8578, pages 213–227. Springer, 2014.

L. Demey and H. Smessaert. The interaction between logic and geometry in Aristotelian diagrams. In M. Jamnik, Y. Uesaka, and S. Elzer Schwartz, editors, *Diagrammatic Representation and Inference*, LNCS 9781, pages 67–82. Springer, 2016a.

L. Demey and H. Smessaert. Metalogical decorations of logical diagrams. *Logica Universalis*, 10:233–292, 2016b.

L. Demey and H. Smessaert. Geometric and cognitive differences between Aristotelian diagrams for the Boolean algebra $\mathbb{B}_4$. *Annals of Mathematics and Artificial Intelligence*, 83:185–208, 2018a.

L. Demey and H. Smessaert. Combinatorial bitstring semantics for arbitrary logical fragments. *Journal of Philosophical Logic*, 47:325–363, 2018b.

T. Denœux, D. Dubois, and H. Prade. Representations of uncertainty in artificial intelligence: Probability and possibility. In P. Marquis, O. Papini, and H. Prade, editors, *A Guided Tour of Artificial Intelligence Research: Volume I: Knowledge Representation, Reasoning and Learning*, pages 69–117. Springer, 2020.

J. Dopp. *Leçons de logique formelle. Première partie: Logique ancienne, La logique des jugements prédicatifs*. Editions de l'Institut Supérieur de Philosophie, 1949.

J. Dopp. *Formal Logic*. Joseph F. Wagner, Inc., 1960.

J. J. Doyle. The hexagon of relationships. *The Modern Schoolman*, 29:93–97, 1952.

D. Dubois and H. Prade. From Blanché's hexagonal organization of concepts to formal concept analysis and possibility theory. *Logica Universalis*, 6:149–169, 2012.

D. Dubois and H. Prade. Gradual structures of oppositions. In L. Magdalena, J. L. Verdegay, and F. Esteva, editors, *Enric Trillas: A Passion for Fuzzy Sets*, pages 79–91. Springer, 2015a.

D. Dubois and H. Prade. Formal concept analysis from the standpoint of possibility theory. In J. Baixeries, C. Sacarea, and M. Ojeda-Aciego, editors, *Formal Concept Analysis (ICFCA 2015)*, LNCS 9113, pages 21–38. Springer, 2015b.

D. Dubois, H. Prade, and A. Rico. The cube of opposition and the complete appraisal of situations by means of Sugeno integrals. In F. Esposito et al., editors, *Foundations of Intelligent Systems (ISMIS 2015)*, LNAI 9384, pages 197–207. Springer, 2015a.

D. Dubois, H. Prade, and A. Rico. The cube of opposition – a structure underlying many knowledge representation formalisms. In Q. Yang and M. Wooldridge, editors, *Proceedings of the Twenty-Fourth International Joint Conference on Artificial Intelligence (IJCAI 2015)*, pages 2933–2939. AAAI Press, 2015b.

D. Dubois, H. Prade, and A. Rico. Graded cubes of opposition and possibility theory with fuzzy events. *International Journal of Approximate Reasoning*, 84:168–185, 2017a.

D. Dubois, H. Prade, and A. Rico. Organizing families of aggregation operators into a cube of opposition. In J. Kacprzyk, D. Filev, and G. Beliakov, editors, *Granular, Soft and Fuzzy Approaches for Intelligent Systems*, pages 27–45. Springer, 2017b.

D. Dubois, H. Prade, and A. Rico. Structures of opposition and comparisons: Boolean and gradual cases. *Logica Universalis*, 14:115–149, 2020.

S. Frijters and L. Demey. The modal logic of Aristotelian diagrams. *Axioms*, 12(471): 1–26, 2023.

S. Furs. Computation of Aristotle's and Gergonne's syllogisms. *Studia Logica*, 46: 209–225, 1987.

J. D. García-Cruz. From the square to octahedra. In J.-Y. Béziau and G. Basti, editors, *The Square of Opposition: A Cornerstone of Thought*, pages 253–272. Springer, 2017.

S. Givant and P. Halmos. *Introduction to Boolean Algebras*. Springer, 2009.

P. V. Grosjean. Théorie algébrique du syllogisme catégorique. *Logique et Analyse*, 15: 547–568, 1972.

E. A. Hacker. The octagon of opposition. *Notre Dame Journal of Formal Logic*, 16: 352–353, 1975.

C. L. Hamblin. *Elementary Formal Logic. A Programmed Course*. Methuen, 1967.

A.-S. Heinemann and L. Demey. Augustus de Morgan's unpublished octagon of opposition. In J.-Y. Beziau and I. Vandoulakis, editors, *The Exoteric Square of Opposition*, pages 141–169. Springer, 2022.

W. Hodges. Two early Arabic applications of model-theoretic consequence. *Logica Universalis*, 12:37–54, 2018.

W. H. Holliday, T. Hoshi, and T. F. Icard III. Information dynamics and uniform substitution. *Synthese*, 190:31–55, 2013.

L. Humberstone. Logical relations. *Philosophical Perspectives*, 27:175–230, 2013.

J. Joerden. On the logic of supererogation. *Jahrbuch für Recht und Ethik*, 6:145–159, 1998.

J. Joerden. Deontological square, hexagon, and decagon: A deontic framework for supererogation. *Logica Universalis*, 6:201–216, 2012.

F. Johnson. Extended Gergonne syllogisms. *Journal of Philosophical Logic*, 26:553–567, 1997.

W. Johnson. *Logic. Part I.* Cambridge University Press, 1921.

R. E. Jones. Truth and contradiction in Aristotle's *De Interpretatione* 6–9. *Phronesis*, 55:26–67, 2010.

S. Kanger. New foundations for ethical theory. In R. Hilpinen, editor, *Deontic Logic: Introductory and Systematic Readings*, pages 36–58. Reidel, 1971.

R. Kennedy. Lemmon on logical relations. *Analysis*, pages 89–93, 1985.

J. N. Keynes. *Studies and Exercises in Formal Logic (Third Edition)*. MacMillan, 1894.

J. N. Keynes. *Studies and Exercises in Formal Logic (Fourth Edition)*. MacMillan, 1906.

E. J. Lemmon. *Beginning Logic*. Thomas Nelson and Sons Ltd, 1965.

T. Libert. Hypercubes of duality. In J.-Y. Béziau and D. Jacquette, editors, *Around and Beyond the Square of Opposition*, pages 293–301. Springer, 2012.

L. Lindahl. *Position and Change. A Study in Law and Logic*. Reidel, 1977.

D. Londey and C. Johanson. Apuleius and the square of opposition. *Phronesis*, 29: 165–173, 1984.

D. Łukasiewicz. The Brentanian inspirations in Tadeusz Czeżowski's philosophy. In A. Brożek, F. Stadler, and J. Woleński, editors, *The Significance of the Lvov-Warsaw School in the European Culture*, pages 79–92. Springer, 2017.

D. Makinson. On the formal representation of rights relations. *Journal of Philosophical Logic*, 15:403–425, 1986.

P. Marquis, O. Papini, and H. Prade. Elements for a history of artificial intelligence. In P. Marquis, O. Papini, and H. Prade, editors, *A Guided Tour of Artificial Intelligence Research: Volume I: Knowledge Representation, Reasoning and Learning*, pages 1–43. Springer, 2020.

J. McNamara. Doing well enough: Toward a logic for common-sense morality. *Studia Logica*, 57:167–192, 1996a.

J. McNamara. Making room for going beyond the call. *Mind*, 105:415–450, 1996b.

A. Menne. *Logik und Existenz*. Westkulturverlag Anton Hain, 1954.

A. Menne, editor. *Logico-Philosophical Studies*. Reidel, 1962.

A. Moktefi and F. Schang. Another side of categorical propositions: The Keynes-Johnson octagon of oppositions. *History and Philosophy of Logic*, 44:459–475, 2023.

A. Moktefi and S.-J. Shin. A history of logic diagrams. In D. M. Gabbay, F. J. Pelletier, and J. Woods, editors, *Handbook of the History of Logic. Volume 11. Logic: A History of its Central Concepts*, pages 611–682. North-Holland, 2012.

A. Moretti. *The Geometry of Logical Opposition*. PhD thesis, University of Neuchâtel, 2009.

T. Parsons. The development of supposition theory in the later 12th through 14th centuries. In D. M. Gabbay and J. Woods, editors, *Handbook of the History of Logic. Volume 2: Mediaeval and Renaissance Logic*, pages 157–280. North-Holland, 2008.

T. Parsons. *Articulating Medieval Logic*. Oxford University Press, 2014.

N. Pfeifer and G. Sanfilippo. Probabilistic squares and hexagons of opposition under coherence. *International Journal of Approximate Reasoning*, 88:282–294, 2017.

C. Pizzi. Generalization and composition of modal squares of opposition. *Logica Universalis*, 10:313–325, 2016.

A. N. Prior. The logic of negative terms in Boethius. *Franciscan Studies*, 13:1–6, 1953.

A. N. Prior. *Formal Logic*. Clarendon Press, 1955.

V. Punčochář. Logical forms, substitutions and information types. *Logic and Logical Philosophy*, 32:459–473, 2023.

J. Raab. The relationship of QUARC and classical logic. Master's thesis, Ludwig-

Maximilians-Universität München, 2016.

H. Reichenbach. The syllogism revised. *Philosophy of Science*, 19:1–16, 1952.

K. Roelandt. *Most or the Art of Compositionality. Dutch de/het meeste at the Syntax-Semantics Interface*. LOT Publications, 2016.

T. Rose. The contradictory function. *Mind*, 66:331–350, 1957.

Z. Rybaříková. Prior on Aristotle's logical squares. *Synthese*, 193:3473–3482, 2016.

P. Sauriol. La structure tétrahexaédrique du système complet ses propositions catégoriques. *Dialogue*, 15:479–501, 1976.

G. Schurz. Rudolf Carnap's modal logic. In W. Stelzner and M. Stöckler, editors, *Zwischen traditioneller und moderner Logik. Nicht-klassische Ansätze*, pages 365–380. Mentis, 2001.

M. Sergot. A computational theory of normative positions. *ACM Transactions on Computational Logic*, 2:581–622, 2001.

M. Sergot. Normative positions. In D. Gabbay, J. Horty, X. Parent, R. van der Meyden, and L. van der Torre, editors, *Handbook of Deontic Logic and Normative Systems*, pages 353–406. College Publications, 2013.

H. Smessaert and L. Demey. Logical geometries and information in the square of opposition. *Journal of Logic, Language and Information*, 23:527–565, 2014.

H. Smessaert and L. Demey. Aristotelian diagrams for the proportional quantifier 'most'. *Axioms*, 12:1–13, 2023.

L. S. Stebbing. *A Modern Introduction to Logic*. Methuen, 1930.

L. S. Stebbing. *Logic in Practice*. Methuen, 1934.

I. Thomas. CS(n): an extension of CS. *Dominican Studies*, 2:145–160, 1949. Reprinted in Menne (1962, p. 40–54).

H. P. van Ditmarsch, W. van der Hoek, and B. P. Kooi. *Dynamic Epistemic Logic*. Springer, 2007.

F. Veltman. *Logics for Conditionals*. PhD thesis, University of Amsterdam, 1985.

J. Wilkinson Miller. *The Structure of Aristotelian Logic*. Kegan Paul, 1938.