Algebraic and Cognitive Aspects of Presenting Aristotelian Diagrams

Lorenz Demey and Hans Smessaert
Structure of the talk

1. Introduction
2. Preliminaries
3. Diagrams with 4 formulas
4. Diagrams with 6 formulas
5. Conclusion
Presenting Aristotelian Diagrams – L. Demey & H. Smessaert
recent years: renewed theoretical interest in the square of oppositions
  • extensions: e.g. hexagons, octagons, cubes, RDHs, etc.
  • families: e.g. Sesmat-Blanché hexagon vs. Sherwood-Czezowski hexagon
  • interrelations: e.g. 6 Sesmat-Blanché hexagons embedded inside RDH

⇒ strong identification between diagram and its formulas:
  1 Aristotelian diagram ↔ 1 set of formulas

different sets of formulas for 1 Aristotelian diagram
  • case studies on different decorations (< logical systems, lexical fields)
  • e.g. Sesmat-Blanché hexagon for modal logic vs. subjective quantification

different Aristotelian diagrams for 1 set of formulas (⇒ our focus today)
  • e.g. set of 4 formulas: square with subalternations \( \downarrow \downarrow \) vs. \( \Rightarrow \) vs. \( \uparrow \uparrow \) vs. \( \iff \)
  • e.g. set of 6 formulas: hexagon (2D) vs. octahedron (3D)
  • e.g. set of 8 formulas: octagon (2D) vs. cube (3D)
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\( \sigma \)-diagrams

- **Logical requirement:** diagrams that are closed under negation
  \( \Rightarrow \) set of \( 2n \) formulas = set of \( n \) pairs of contradictory formulas (PCDs)

- **Geometrical requirement:** PCDs share a central symmetry point

- **Nearly all Arist. diagrams in the literature satisfy these requirements**

- **Example:** Sesmat-Blanché hexagon for modal logic S5
Abstract configurations of PCDs

- number of configurations of $n$ PCDs: $n! \cdot 2^n$
  - $n!$ permutations of the $n$ PCDs
  - each of the $n$ PCDs can be put into the configuration in 2 ways:
    $$(\varphi, \neg \varphi) \text{ or } (\neg \varphi, \varphi) \Rightarrow 2 \times 2 \times \cdots \times 2 = 2^n$$
- independent of concrete geometric visualization
- some concrete examples:
  - $n = 2$: $2! \cdot 2^2 = 2 \cdot 4 = 8$
  - $n = 3$: $3! \cdot 2^3 = 6 \cdot 8 = 48$
  - $n = 4$: $4! \cdot 2^4 = 24 \cdot 16 = 384$
every diagram has a number of symmetries
(= cardinality of its symmetry group)

visualize \( n \) PCDs using a diagram \( D \) that has \( k \) symmetries
\[ \Rightarrow \frac{n! \cdot 2^n}{k} \] fundamentally distinct presentations of \( D \)

two presentations are fundamentally distinct iff
one cannot be obtained by reflecting and/or rotating the other

general aim:

fundamental geometrical differences should correspond
(as much as possible) with logical differences
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given is a fixed set of 4 formulas, i.e. 2 PCDs
⇒ \(2! \cdot 2^2 = 8\) abstract configurations

visualize these 8 abstract configurations using ordinary squares
these 8 squares are symmetric and/or rotational variants of each other

this should not be surprising:
- symmetry group of the square: dihedral group $D_4$
- $D_4$ has 8 elements

$$\frac{2! \cdot 2^2}{|D_4|} = \frac{8}{8} = 1$$ fundamental presentation

there is exactly one way (up to symmetries and rotations) of visualizing 4 formulas using a square
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Diagrams with 6 formulas

- fixed set of 6 formulas/3 PCDs $\Rightarrow 3! \cdot 2^3 = 48$ abstract configurations
- visualize these 48 abstract configurations using 48 **hexagons**
  - symmetry group of the hexagon: $D_6$: 12 symmetries
    $\Rightarrow \frac{48}{12} = 4$ fundamental presentations of the hexagon
- visualize these 48 abstract configurations using 48 **octahedrons**
  - symmetry group of the octahedron: $O_h$: 48 symmetries
    $\Rightarrow \frac{48}{48} = 1$ fundamental presentation of the octahedron
Case study 1: Jacoby-Sesmat-Blanché $\sigma_3$

- 6 formulas: $\Box p, \Box \neg p, \Diamond p, \Diamond \neg p, \Box p \lor \Box \neg p, \Diamond p \land \Diamond \neg p$

- 4 fundamental presentations of the hexagon:

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Case study 1: Jacoby-Sesmat-Blanché $\sigma_3$

- 6 formulas: $\square p, \square \neg p, \lozenge p, \lozenge \neg p, \square p \lor \square \neg p, \lozenge p \land \lozenge \neg p$
- 4 fundamental presentations of the hexagon:
Case study 2: fully degenerated $\sigma_3$

- 6 formulas:
  
  \[ p, \neg p, \Box p \lor \neg \Diamond p, \neg \Box p \land \Diamond p, \Box p \lor (\neg p \land \Diamond p), \neg \Diamond p \lor (p \land \neg \Box p) \]

- fully degenerated:
  
  - \( \frac{6 \cdot 5}{2} = \frac{30}{2} = 15 \) pairs of formulas
  - 3 pairs: contradictory (\( \Rightarrow 3 \) PCDs \( \Rightarrow \sigma_3 \))
  - 12 other pairs: no Aristotelian relation whatsoever (unconnectedness)

- 4 fundamental presentations of the hexagon:

  - no differences between the 4 presentations whatsoever!
• Jacoby-Sesmat-Blanché $\sigma_3$
  • 4 fundamental presentations of the hexagon: geometrical differences
  • corresponding logical differences: 1 vs 2,3,4
  \[ \Rightarrow \] hexagon visualization is preferred!

• fully degenerated $\sigma_3$ (12 × unconnectedness)
  • 4 fundamental presentations of the hexagon: geometrical differences
  • no corresponding logical differences whatsoever
  \[ \Rightarrow \] octahedron visualization is preferred!

• what about other types of $\sigma_3$?
  • Sherwood-Czezowksi
  • minimally degenerated (4 × unconnectedness)
  • intermediately degenerated (8 × unconnectedness)
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systematic study of the different diagrams for a fixed set of formulas

general case: $\sigma_n \Rightarrow$ concrete visualizations vs abstract mathematics

a simple two-dimensional regular $2n$-gon has $4n$ symmetries

$$\Rightarrow \frac{n! \cdot 2^n}{4n} = (n - 1)! \cdot 2^{n-2}$$ fundamental presentations

in abstract $n$-dimensional space: cross-polytope

- dual of the $n$-dimensional hypercube
- centrally symmetric polytope with $2n$ vertices and $n! \cdot 2^n$ symmetries

$$\Rightarrow \frac{n! \cdot 2^n}{n! \cdot 2^n} = 1$$ fundamental presentation

concrete illustration: $\sigma_4$ (Buridan, Béziau, Moretti, ...)

- 2D octagon $\frac{4! \cdot 2^4}{4 \cdot 4} = \frac{384}{16} = 24$ fundamental presentations

- 4D 16-cell $\frac{4! \cdot 2^4}{4! \cdot 2^4} = \frac{384}{384} = 1$ fundamental presentation
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- 2D octagon $\frac{4! \cdot 2^4}{4 \cdot 4} = \frac{384}{16} = 24$ fundamental presentations
- 3D cube $\frac{4! \cdot 2^4}{48} = \frac{384}{48} = 8$ fundamental presentations
- 4D 16-cell $\frac{4! \cdot 2^4}{4! \cdot 2^4} = \frac{384}{384} = 1$ fundamental presentation
Thank you!

More info: www.logicalgeometry.org