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The Relationship between Aristotelian and Hasse Diagrams

Lorenz Demey and Hans Smessaert

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Overview

- various families of diagrams used in logic:
  - Aristotelian diagrams
  - Hasse diagrams — 1 diagram $\leftrightsquigarrow$ # formulas
  - duality diagrams
  - Euler diagrams
  - spider diagrams — 1 diagram $\leftrightsquigarrow$ 1 formula
  - Peirce's existential graphs
  - ...
- this talk: focus on Aristotelian diagrams and Hasse diagrams
  - what do these two types of diagrams look like?
  - comparison of the two types
  - a unified account: visual-cognitive aspects
  - a unified account: logico-geometrical aspects

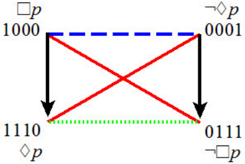
- the Aristotelian relations: φ and ψ are

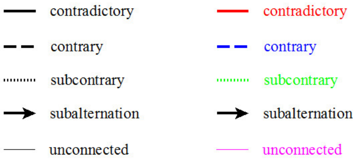
contradictory	iff	$\models \neg(\varphi \wedge \psi)$	and	$\models \neg(\neg\varphi \wedge \neg\psi)$
contrary	iff	$\models \neg(\varphi \wedge \psi)$	and	$\not\models \neg(\neg\varphi \wedge \neg\psi)$
subcontrary	iff	$\not\models \neg(\varphi \wedge \psi)$	and	$\models \neg(\neg\varphi \wedge \neg\psi)$
in subalternation	iff	$\models \varphi \rightarrow \psi$	and	$\not\models \psi \rightarrow \varphi$

- almost all Aristotelian diagrams in the literature satisfy the following:
  - the formulas are *contingent*
  - the formulas are *pairwise non-equivalent*
  - the formulas come in *contradictory pairs* (φ—$\neg\varphi$)
  - these pairs are ordered around a *center of symmetry*

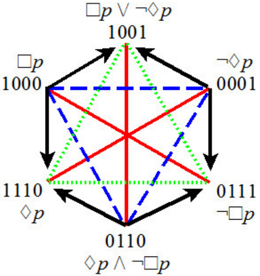
- Aristotelian diagrams in logic:
  - very long and rich tradition (Aristotle/Apuleius)
  - contemporary logic: lingua franca to talk about logical systems (modal logic, epistemic logic, dynamic logic, deontic logic, etc.)

classical square for S5

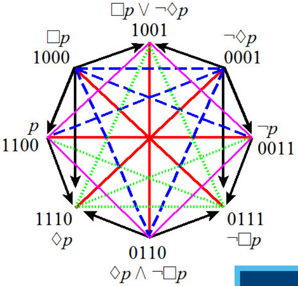




Jacoby-Sesmat-Blanché hexagon



Béziau octagon



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- a Hasse diagram visualizes a partially ordered set $(P, \leq)$:

$\leq$ is reflexive:	for all $x \in P : x \leq x$
$\leq$ is transitive:	for all $x, y, z \in P : x \leq y, y \leq z \Rightarrow x \leq z$
$\leq$ is antisymmetric:	for all $x, y \in P : x \leq y, y \leq x \Rightarrow x = y$

- Hasse diagrams in logic and mathematics:

divisibility poset	$x \leq y$ iff x divides y
subgroup lattices	$x \leq y$ iff x is a subgroup of y
logic/Boolean algebra	$x \leq y$ iff x logically entails y

- we focus on Boolean algebras
  - always have a Hasse diagram that is centrally symmetric
  - can be partitioned into 'levels' $L_0, L_1, L_2, \ldots$
    - $L_0 = \{\bot\}$
    - $L_{i+1} = \{y \mid \exists x \in L_i : x \lessdot y\}$
    - for all $x, y \in L_i : x \not\leq y$ and $y \not\leq x$

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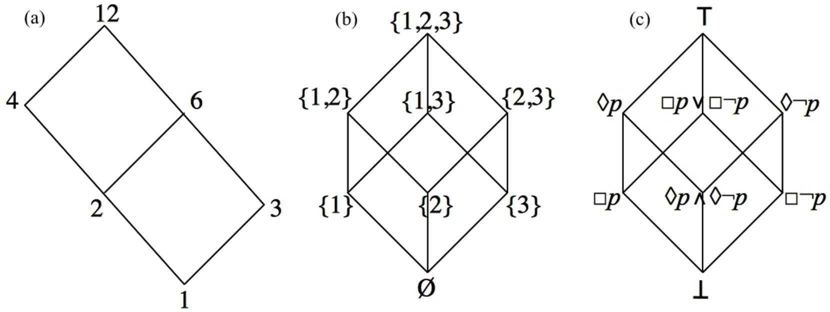
Some examples:

(a) the divisors of 12
(b) the Boolean algebra $\wp(\{1, 2, 3\})$
(c) a Boolean algebra of formulas from the modal logic S5



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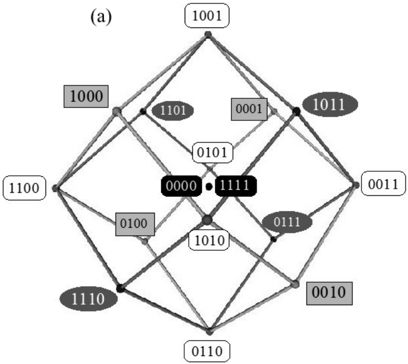
- recent years: move toward 3D diagrams
- example: rhombic dodecahedron
  - (a) as an Aristotelian diagram (Moretti, Smessaert, etc.)
  - (b) as a Hasse diagram (Zellweger, Kauffman, etc.)



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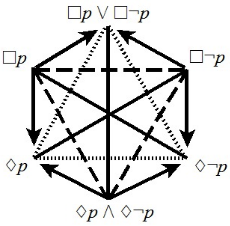
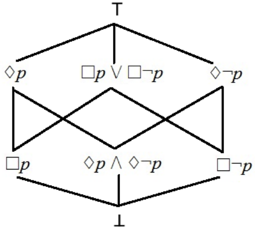
- 3 differences
  1. the non-contingent formulas $\bot$ and $\top$
  2. the general direction of the entailments
  3. visualization of the levels
- the non-contingent formulas $\bot$ and $\top$
  - Hasse diagrams: begin- and endpoint of the $\leq$-ordering
  - Aristotelian diagrams: $\bot$ and $\top$ usually *not* visualized
  - Sauriol, Smessaert, etc.: $\bot$ and $\top$ coincide in the center of symmetry





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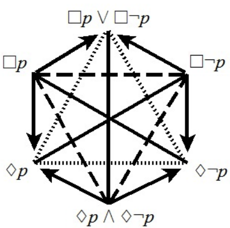
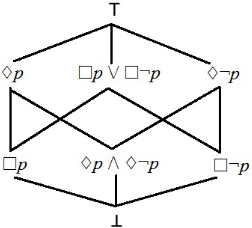
- the general direction of the entailments
  - Hasse diagrams: all entailments go upwards
  - Aristotelian diagrams: no single shared direction
- visualization of the levels
  - Hasse diagrams: levels L_i are visualized as horizontal hyperplanes
  - Aristotelian diagrams: no uniform visualization of levels





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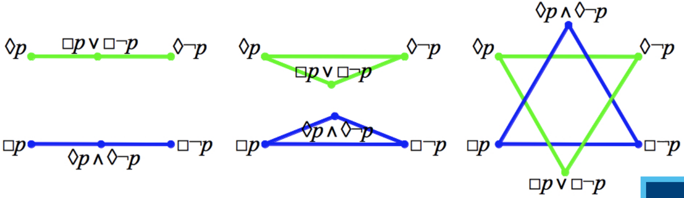
- dissimilarities explained by general cognitive principles (Tversky et al.):
  - **Congruity Principle**: content/structure of visualization correspond to content/structure of desired mental representation
  - **Apprehension Principle**: content/structure of visualization are readily and correctly perceived and understood
  - information selection/omision and simplification/distortion
- different visual properties $\leftrightsquigarrow$ different goals
  - Aristotelian diagrams: visualize the Aristotelian relations
  - Hasse diagrams: visualize the structure of the entailment ordering $\leq$
- Hasse diagrams: strong congruence between logical & visual properties
  - shared direction of entailment (vertically upward)
  - levels as horizontal lines/planes
    - if $\varphi, \psi \in L_i$, then $\varphi \nleq \psi$ and $\psi \nleq \varphi$
    - formulas of a single level are *independent* of each other w.r.t. $\leq$
    - level = horizontal $\Rightarrow$ *orthogonal* to the vertical $\leq$-direction

- consider the three S5-formulas $\Box p$, $\Box\neg p$, $\Diamond p \wedge \Diamond\neg p$
  - Hasse perspective: all belong to L_1 $\Rightarrow$ horizontal line
  - Aristotelian perspective: all contrary to each other
- the contrariety between $\Box p$ and $\Box\neg p$ overlaps with the two others
  - serious violation of the apprehension principle
  - direct reason: the three formulas lie on a single line
- this is solved in the Aristotelian diagram:
  - move $\Diamond p \wedge \Diamond\neg p$ away from the line between $\Box p$ and $\Box\neg p$
  - triangle of contrarieties $\Rightarrow$ in line with apprehension principle
  - mixing of levels, no single entailment direction, $\bot$ moves to middle



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- we restrict ourselves to Aristotelian diagrams that are Boolean closed
  - Boolean closed: JSB hexagon, RDH, ...
  - not Boolean closed: the square, the Béziau octagon, ...
- this is no substantial restriction
  - every Aristotelian diagram embeds into one that *is* Boolean closed
  - the square embeds into JSB, the Béziau octagon embeds into RDH, ...
- this presentation: intuitive explanations, low-dimensional examples
  - mathematical detail
  - full generality ⇒ see the paper
  - high-dimensional cases

- consider the Boolean algebra $\mathbb{B}_3$
  - $\mathbb{B}_3$ has $2^3 = 8$ elements
  - elements: e.g. formulas of the modal logic S5
  - canonical representation: bitstrings

$\bot$	$\Box p$	$\Diamond p \wedge \Diamond\neg p$	$\Box\neg p$	$\Diamond\neg p$	$\Box p \vee \Box\neg p$	$\Diamond p$	$\top$
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
000	100	010	001	011	101	110	111

- the Hasse diagram of $\mathbb{B}_3$ can be drawn as a three-dimensional cube
  - general entailment direction runs from 000 to 111
  - logical levels $\leftrightsquigarrow$ planes orthogonal to the entailment direction

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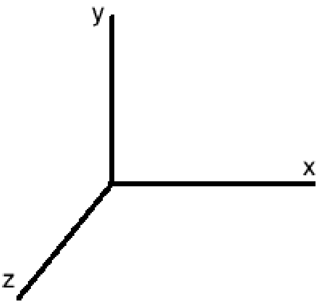
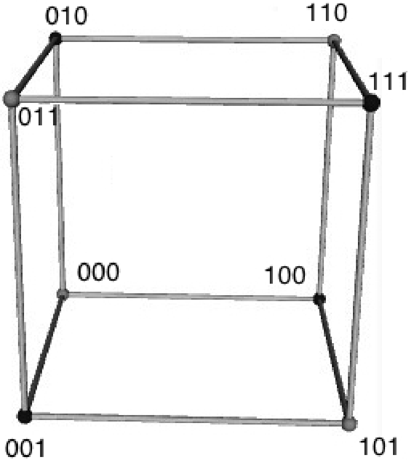
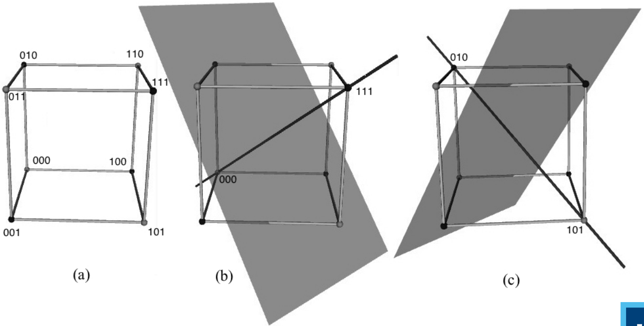

y
x
z

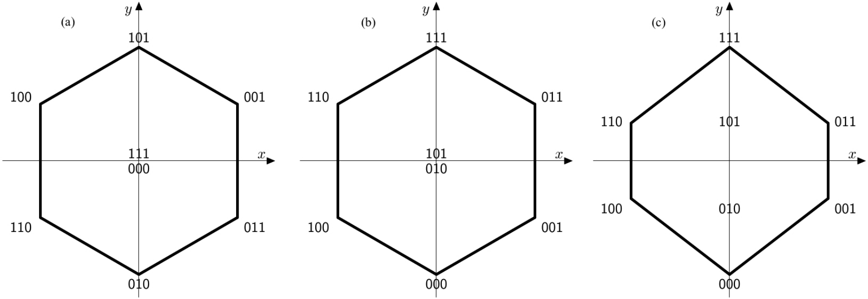

010
110
111
011
000
100
001
101

- in (a) the cube consists of 4 pairs of diametrically opposed vertices:
  - 3 contingent pairs: 101—010, 110—001, 011—100
  - 1 non-contingent pair: 111—000
  - each pair defines a projection axis for a **vertex-first projection**:
- in (b) projection along 111—000 axis
- in (c) projection along 101—010 axis



(a) (b) (c)

- the vertex-first projections from 3D cube to 2D hexagon:
  - (a) projection along 111—000 ⇒ Aristotelian diagram (JSB)
  - (b) projection along 101—010 ⇒ Hasse diagram (almost)
- if we slightly 'nudge' the projection axis 101—010, we get:
  - (c) projection 'along' 101—010 ⇒ Hasse diagram

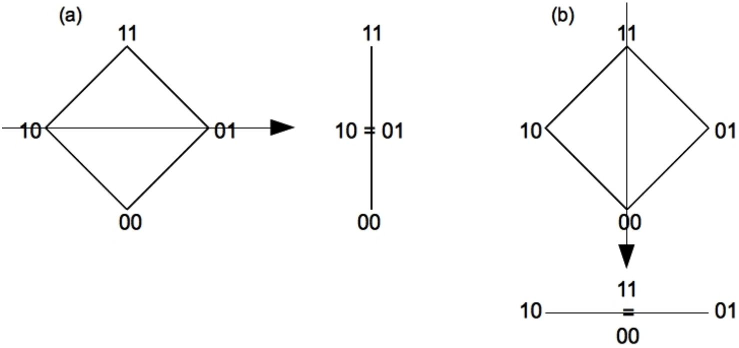


- Aristotelian and Hasse diagram: both vertex-first projections of cube
  - Aristotelian diagram: project along the entailment direction
  - Hasse diagram: project along another direction
- recall the dissimilarities between Aristotelian and Hasse diagrams:
  1. the position of $\bot$ and $\top$
  2. the general direction of the entailments
  3. the visualization of the levels
- these three differences turn out to be interrelated:
  different manifestations of a single choice (projection direction)
- now illustrate these differences by means of more basic vertex-first projections from 2D square to 1D line

difference 1: the position of $\bot$ and $\top$

the square is a Hasse diagram $\Rightarrow$ $\bot$ and $\top$ as lowest and highest point

(a) project along other direction $\Rightarrow$ $\bot$ and $\top$ still as lowest and highest
(b) project along the $\top/\bot$ direction $\Rightarrow$ $\bot$ and $\top$ coincide in the center



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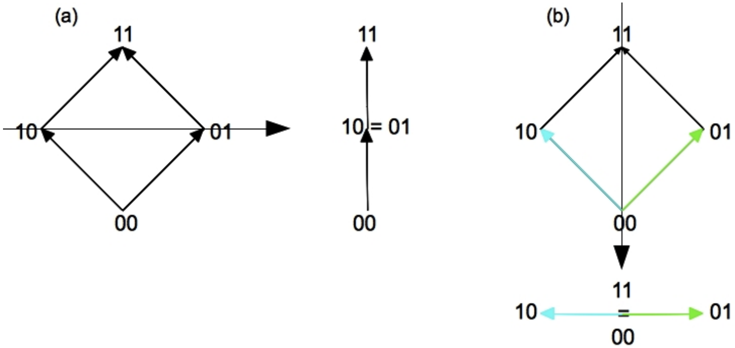
difference 2: the general direction of the entailments

the square is a Hasse diagram ⇒ general entailment direction is upwards

(a) project along other direction ⇒ general entailment direction is still upwards
(b) project along the ⊤/⊥ direction ⇒ no longer a general entailment direction



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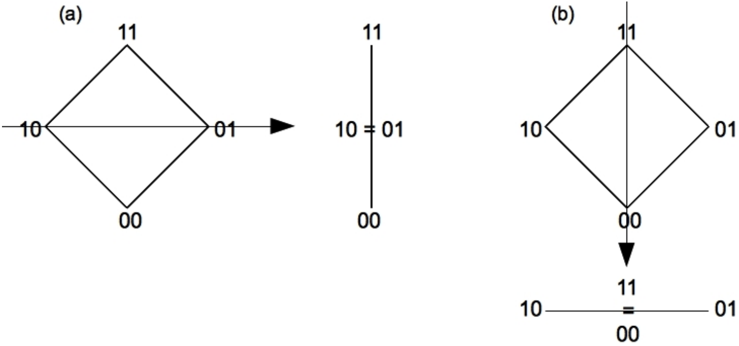
difference 3: the visualization of the levels

the square is a Hasse diagram $\Rightarrow$ uniform (horizontal) levels

(a) project along other direction $\Rightarrow$ still uniform (horizontal) levels
(b) project along the $\top/\bot$ direction $\Rightarrow$ mixing of levels



- visual differences between Aristotelian and Hasse diagrams
  - the issue of $\bot$ and $\top$ (present/absent)
  - the general direction of the entailments (shared/not shared)
  - the visualization of the levels (uniform/mixed)

- unified account of Aristotelian and Hasse diagrams
  - cognitive part: different visual properties
    $\Leftarrow$ different visualization strategies
    $\Leftarrow$ different goals
  - geometrical part: three types of visual differences
    $=$ three manifestations of a single choice:

Aristotelian diagram $\Leftarrow$ vertex-first projection along $\top/\bot$ direction

Hasse diagram $\Leftarrow$ vertex-first projection along another direction

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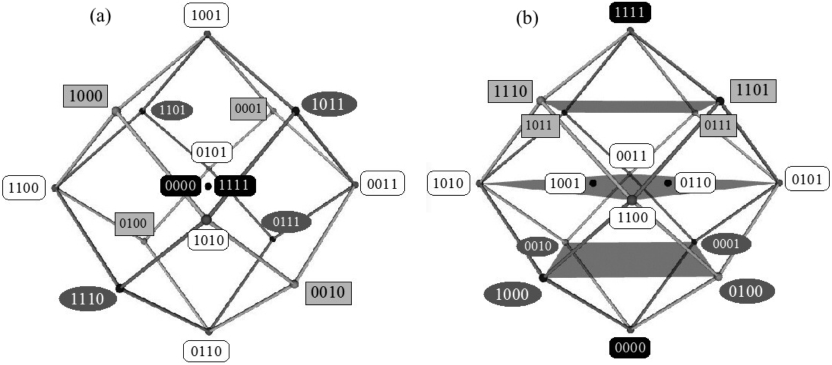
Conclusion

- generalization of the vertex-first projections:
  - from 2D square to 1D line
  - from 3D cube to 2D hexagon
  - from 4D hypercube to 3D **rhombic dodecahedron**



rhombic dodecahedron RDH

Thank you!

More info: www.logicalgeometry.org