KU LEUVEN

Metalogical Decorations of Logical Diagrams

Lorenz Demey & Hans Smessaert

CLMPS 2015, Helsinki

- Aristotelian diagrams (e.g. square of oppositions):
  - long and rich history in philosophical logic
  - past decade: revived interest
  - mainly object-logical decorations: formulas from some logical system
  - some exceptions: metalogical decorations (Béziau, Seuren)
- aims of this talk:
  - extend and deepen our knowledge of metalogical decorations
  - new metalogical decorations, larger diagrams, less well-known diagrams
  - unifiying perspective on existing work
- keep in mind:
  - this talk is based on a paper of 60+ pages
  - omission of many details, examples, etc.
  - interested? ask for the full paper!

- the Aristotelian relations (in a suitable logical system S): φ and ψ are

S-contradictory	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \models \neg(\neg\varphi \wedge \neg\psi)$
S-contrary	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \not\models \neg(\neg\varphi \wedge \neg\psi)$
S-subcontrary	iff	$\mathsf{S} \not\models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \models \neg(\neg\varphi \wedge \neg\psi)$
in S-subalternation	iff	$\mathsf{S} \models \varphi \rightarrow \psi$	and	$\mathsf{S} \not\models \psi \rightarrow \varphi$

- this can be generalized to an arbitrary Boolean algebra $\mathbb{B}$: x and y are

$\mathbb{B}$-contradictory	iff	$x \wedge_{\mathbb{B}} y = \bot_{\mathbb{B}}$	and	$x \vee_{\mathbb{B}} y = \top_{\mathbb{B}}$
$\mathbb{B}$-contrary	iff	$x \wedge_{\mathbb{B}} y = \bot_{\mathbb{B}}$	and	$x \vee_{\mathbb{B}} y \neq \top_{\mathbb{B}}$
$\mathbb{B}$-subcontrary	iff	$x \wedge_{\mathbb{B}} y \neq \bot_{\mathbb{B}}$	and	$x \vee_{\mathbb{B}} y = \top_{\mathbb{B}}$
in $\mathbb{B}$-subalternation	iff	$x \wedge_{\mathbb{B}} y = x$	and	$x \wedge_{\mathbb{B}} y \neq y$

- this subsumes both object- and metalogical uses:
  - object-logical: let $\mathbb{B}$ be $\mathbb{B}(\mathsf{S})$ (Lindenbaum-Tarski algebra of S)
  - metalogical: let $\mathbb{B}$ be $\wp(\mathbb{B}(\mathsf{S}))$ or $\wp(\mathbb{B}(\mathsf{S}) \times \mathbb{B}(\mathsf{S}))$

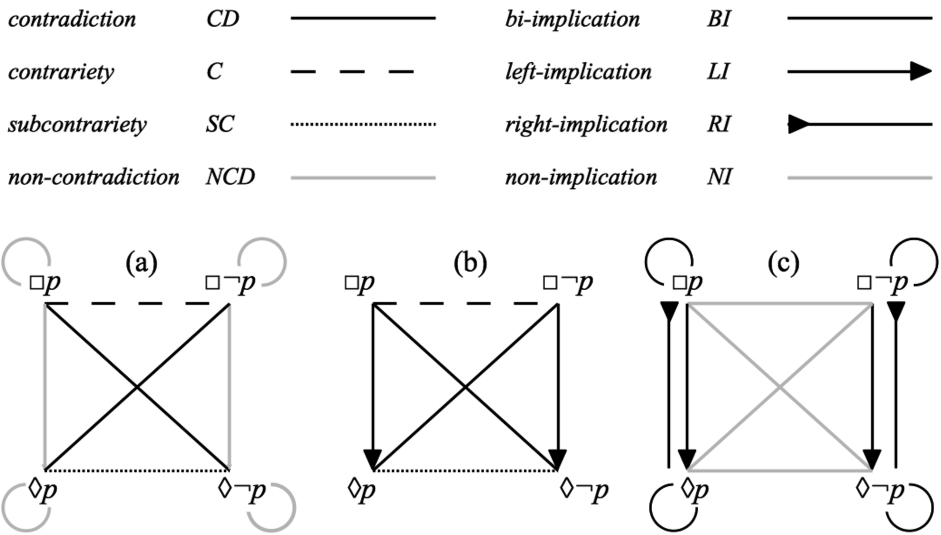
- the opposition relations: φ and ψ are

S-contradictory	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \models \neg(\varphi \wedge \psi)$
S-contrary	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \not\models \neg(\varphi \wedge \psi)$
S-subcontrary	iff	$\mathsf{S} \not\models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \models \neg(\varphi \wedge \psi)$
S-noncontradictory	iff	$\mathsf{S} \not\models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \not\models \neg(\varphi \wedge \psi)$

- the implication relations: φ and ψ are

in S-bi-implication	iff	$\mathsf{S} \models \varphi \rightarrow \psi$	and	$\mathsf{S} \models \psi \rightarrow \varphi$
in S-left-implication	iff	$\mathsf{S} \models \varphi \rightarrow \psi$	and	$\mathsf{S} \not\models \psi \rightarrow \varphi$
in S-right-implication	iff	$\mathsf{S} \not\models \varphi \rightarrow \psi$	and	$\mathsf{S} \models \psi \rightarrow \varphi$
in S-non-implication	iff	$\mathsf{S} \not\models \varphi \rightarrow \psi$	and	$\mathsf{S} \not\models \psi \rightarrow \varphi$

- motivation:
  - disentangling the Aristotelian relations into opposition and implication
  - the Aristotelian relations are informationally optimal between the opposition and implication relations



contradiction
CD
contrariety
C
subcontrariety
SC
non-contradiction
NCD
bi-implication
BI
left-implication
LI
right-implication
RI
non-implication
NI
(a)
(b)
(c)
□p
□¬p
◊p
◊¬p

- Boolean algebras $\mathbb{A}$ and $\mathbb{B}$
- the duality relations: the n-ary operators $O_1, O_2 \colon \mathbb{A}^n \to \mathbb{B}$ are

identical	iff	$\forall a \in A^n \colon O_1(a) = O_2(a)$
external negations	iff	$\forall a \in A^n \colon O_1(a) = \neg_{\mathbb{B}} O_2(a)$
internal negations	iff	$\forall a \in A^n \colon O_1(a) = O_2(\neg_{\mathbb{A}^n} a)$
duals	iff	$\forall a \in A^n \colon O_1(a) = \neg_{\mathbb{B}} O_2(\neg_{\mathbb{A}^n} a)$

—with $\neg_{\mathbb{A}^n} a = \neg_{\mathbb{A}^n}(a_1, \ldots, a_n) = (\neg_{\mathbb{A}} a_1, \ldots, \neg_{\mathbb{A}} a_n)$

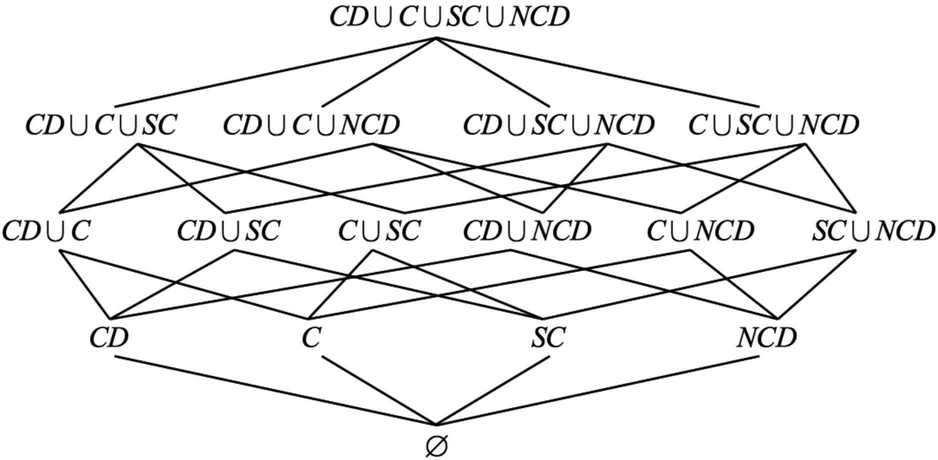
- abbreviations: $\textsc{id}$, $\textsc{eneg}$, $\textsc{ineg}$ and $\textsc{dual}$
- examples: $\textsc{ineg}(\Box, \Box\neg)$, $\textsc{dual}(\Box, \Diamond)$, $\textsc{dual}(\wedge, \vee)$, etc.
- note:
  - many Aristotelian squares are also duality squares
  - but the Aristotelian and duality relations are conceptually independent (except that CD is $\textsc{eneg}$, of course)

- logical system S (often left implicit)

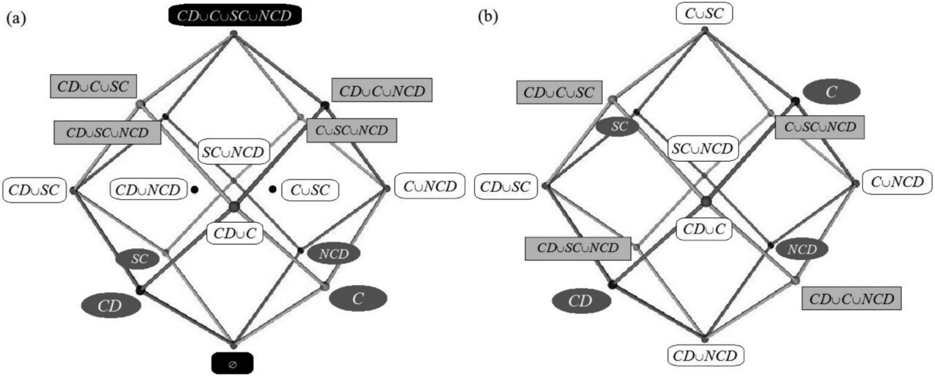
- easy: every pair of formulas stands in exactly one opposition relation
- the opposition relations form a partition of $\mathbb{B}(\mathsf{S}) \times \mathbb{B}(\mathsf{S})$
- the opposition relations can be viewed as atoms in a Boolean algebra
  - the elements of this Boolean algebra are $\bigcup \mathcal{X}$, for $\mathcal{X} \subseteq \{CD, C, SC, NCD\}$
  - it has $2^4 = 16$ elements
  - its bottom and top elements are $\emptyset$ and $CD \cup C \cup SC \cup NCD = \mathbb{B}(\mathsf{S}) \times \mathbb{B}(\mathsf{S})$

- visualizations of this Boolean algebra:
  - Hasse diagram: 2D or 3D rhombic dodecahedron (RDH)
  - Aristotelian diagram: rhombic dodecahedron

  (close connection between Hasse RDH and Aristotelian RDH)


CD ∪ C ∪ SC ∪ NCD
CD ∪ C ∪ SC
CD ∪ C ∪ NCD
CD ∪ SC ∪ NCD
C ∪ SC ∪ NCD
CD ∪ C
CD ∪ SC
C ∪ SC
CD ∪ NCD
C ∪ NCD
SC ∪ NCD
CD
C
SC
NCD
∅

KU LEUVEN


(a)
CD∪C∪SC∪NCD
CD∪C∪SC
CD∪C∪NCD
CD∪SC∪NCD
C∪SC∪NCD
SC∪NCD
CD∪SC
CD∪NCD
C∪SC
C∪NCD
CD∪C
SC
NCD
CD
C
∅
(b)
C∪SC
CD∪C∪SC
C
SC
C∪SC∪NCD
SC∪NCD
CD∪SC
C∪NCD
CD∪C
CD∪SC∪NCD
NCD
CD
CD∪C∪NCD
CD∪NCD

- Aristotelian RDH for the opposition relations
  ⇒ largest metalogical diagram so far!

- Aristotelian RDH has many object-logical decorations
  - e.g. propositional connectives, modal logic S5, subjective quantifiers (many/few), public announcement logic, etc.

- its internal structure has been extensively studied:
  - it contains 4 weak Jacoby-Sesmat-Blanché hexagons (Pellissier)
  - it contains 6 strong JSB hexagons (Béziau, Moretti, HS)
  - it contains 12 Sherwood-Czezowski hexagons (HS, LD)
  - it contains 6 Buridan octagons (HS, LD)
  - complementarity between JSB hexagons and Buridan octagons (HS, LD)

⇒ all these properties straightforwardly carry over
from the object-logical to the metalogical level

- strong and weak notions of (sub)contrariety: φ and ψ are

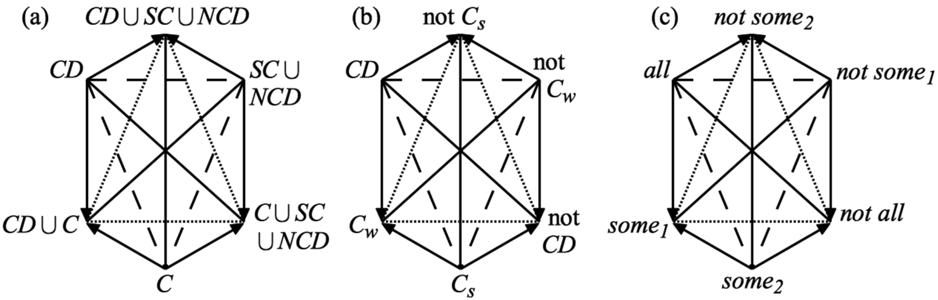
strongly S-*contrary*	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \not\models \varphi \vee \psi$
weakly S-*contrary*	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$		
strongly S-*subcontrary*	iff	$\mathsf{S} \not\models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \models \varphi \vee \psi$
weakly S-*subcontrary*	iff			$\mathsf{S} \models \varphi \vee \psi$

- Humberstone: "traditionalist approach" vs "modernist approach"

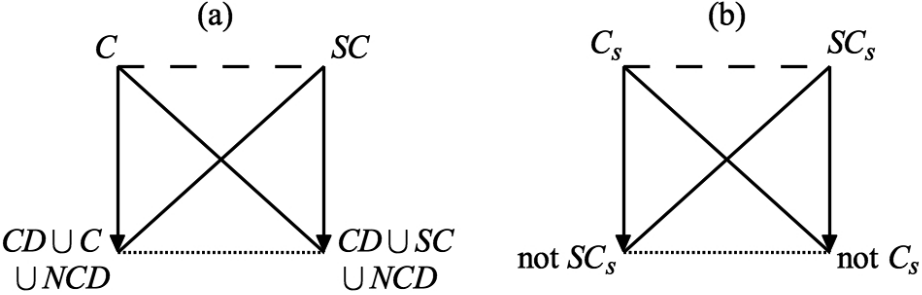
- connection with the opposition relations:

$$C_s = C \qquad SC_s = SC$$
$$C_w = CD \cup C \qquad SC_w = CD \cup SC$$

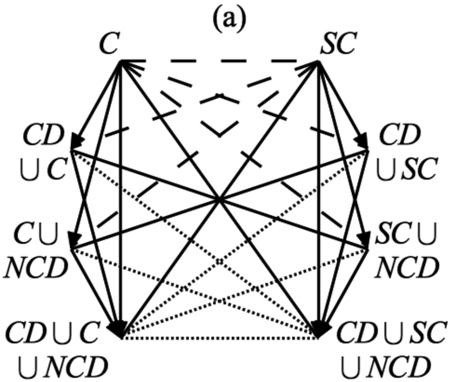
- note that $CD = C_w \cap SC_w$



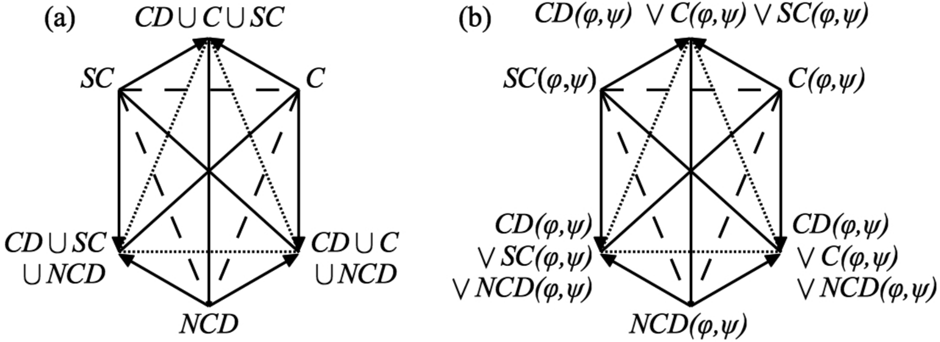
- pragmatic perspective:
  - $\langle CD, C_w \rangle$ forms a Horn scale
  - saying C_w triggers the scalar implicature not-CD
  - total meaning becomes: C_w but not CD, i.e. C_s
- analogy: unilateral and bilateral *some*
  - *at least one* versus *some but not all*



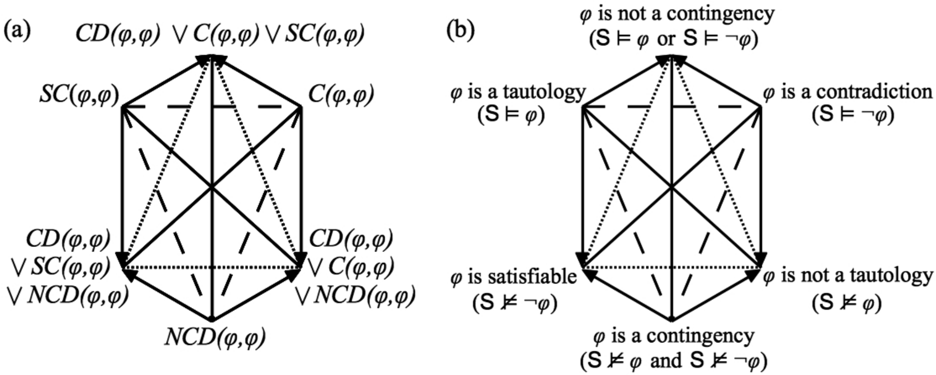
- the subalternation from C_s to not-SC_s can be split up by putting C_w in between
- the subalternation from SC_s to not-C_s can be split up by putting SC_w in between


(a)
C
SC
CD
∪C
CD
∪SC
C∪
NCD
SC∪
NCD
CD∪C
∪NCD
CD∪SC
∪NCD
(b)
C_s
SC_s
C_w
SC_w
C_w^*
SC_w^*
not SC_s
not C_s

- in terms of relations
- in terms of statements about formulas φ, ψ



- what happens if we fill in the same formula twice (i.e. $\varphi = \psi$)?
- we obtain well-known metalogical notions
- this hexagon was first studied by Béziau



KU LEUVEN

- the implication relations closely resemble the opposition relations

$$\begin{array}{rcl} CD(\varphi,\psi) & \text{iff} & BI(\varphi,\neg\psi) \\ C(\varphi,\psi) & \text{iff} & LI(\varphi,\neg\psi) \\ SC(\varphi,\psi) & \text{iff} & RI(\varphi,\neg\psi) \\ NCD(\varphi,\psi) & \text{iff} & NI(\varphi,\neg\psi) \end{array}$$

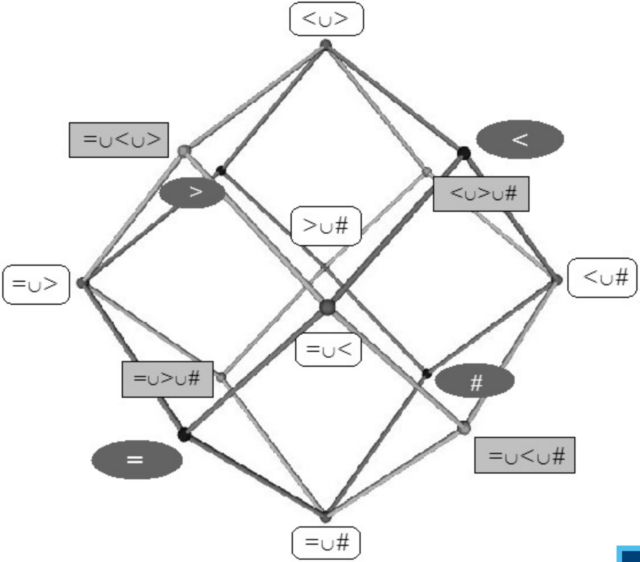
- the implication relations form a partition of $\mathbb{B}(\mathsf{S}) \times \mathbb{B}(\mathsf{S})$

$\Rightarrow$ atoms of a Boolean algebra
$\Rightarrow$ Hasse RDH for this Boolean algebra
$\Rightarrow$ Aristotelian RDH for this Boolean algebra
$\Rightarrow$ study the subdiagrams of this Aristotelian RDH

$\vdots$

- consider an arbitrary partial order $\leq$ on some set X
- some notions:
  - $x < y :\Leftrightarrow (x \leq y$ and $x \neq y)$
  - $x > y :\Leftrightarrow (x \geq y$ and $x \neq y)$
  - $x \# y :\Leftrightarrow$ not$(x < y$ or $x > y)$
- easy to show: $=, <, >, \#$ form a partition of S
- if $\leq$ happens to be the $\models$-relation on $\mathbb{B}(\mathsf{S})$:

$$\begin{array}{lcl} = & \text{corresponds to} & BI \\ < & \text{corresponds to} & LI \\ > & \text{corresponds to} & RI \\ \# & \text{corresponds to} & NI \end{array}$$

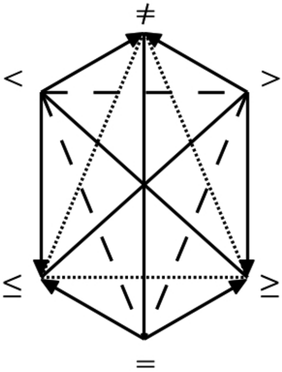


From Partial to Total Orders

- from partial order to total order:
  - impose the additional axiom of totality: $\forall x, y \in S : x \leq y$ or $x \geq y$
  - equivalently, impose the assumption that $\# = \emptyset$
- effect on the Aristotelian RDH: pairwise collapses:

RDH		collapse	collapse		RDH
BI	$\rightarrow$	BI	$LI \cup RI$	$\leftarrow$	$LI \cup RI \cup NI$
$BI \cup NI$	$\nearrow$			$\nwarrow$	$LI \cup RI$
LI	$\rightarrow$	LI	$BI \cup RI$	$\leftarrow$	$BI \cup RI \cup NI$
$LI \cup NI$	$\nearrow$			$\nwarrow$	$BI \cup RI$
RI	$\rightarrow$	RI	$BI \cup LI$	$\leftarrow$	$BI \cup LI \cup NI$
$RI \cup NI$	$\nearrow$			$\nwarrow$	$BI \cup LI$
NI	$\rightarrow$	$[\emptyset]$	$[BI \cup LI \cup RI]$	$\leftarrow$	$BI \cup LI \cup RI$
$[\emptyset]$	$\nearrow$			$\nwarrow$	$[BI \cup LI \cup RI \cup NI]$

- the Aristotelian RDH collapses into a strong JSB hexagon
- this hexagon was already known by Blanché (= the ‘B’ in ‘JSB’)



KU LEUVEN

- Aristotelian = hybrid between opposition/implication
  $\Rightarrow$ some Aristotelian diagrams for opposition/implication relations can also be viewed as Aristotelian diagrams for the Aristotelian relations

  (e.g. Buridan octagon for strong/weak (sub)contrariety)

- but: in each of these diagrams:
  - *either* only opposition relations
  - *or* only implication relations

- now: diagrams that contain both opposition and implication relations

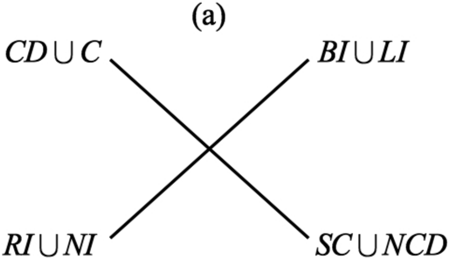
- already in the 80s, Löbner claimed that the following four relations form an Aristotelian square:

compatibility	$\not\models \neg(\varphi \wedge \psi)$
implication	$\models \varphi \rightarrow \psi$
contrariety	$\models \neg(\varphi \wedge \psi)$
non-implication	$\not\models \varphi \rightarrow \psi$

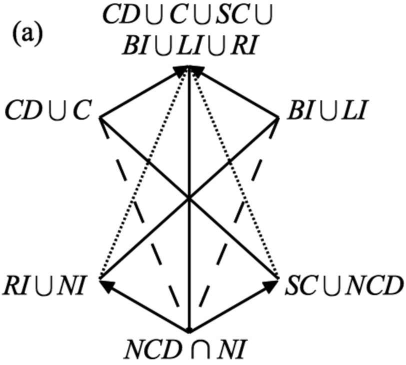
- already in the 80s, Löbner claimed that the following four relations form an Aristotelian square:

compatibility	$\not\models \neg(\varphi \wedge \psi)$	$SC \cup NCD$
implication	$\models \varphi \to \psi$	$BI \cup LI$
contrariety	$\models \neg(\varphi \wedge \psi)$	$CD \cup C$
non-implication	$\not\models \varphi \to \psi$	$RI \cup NI$

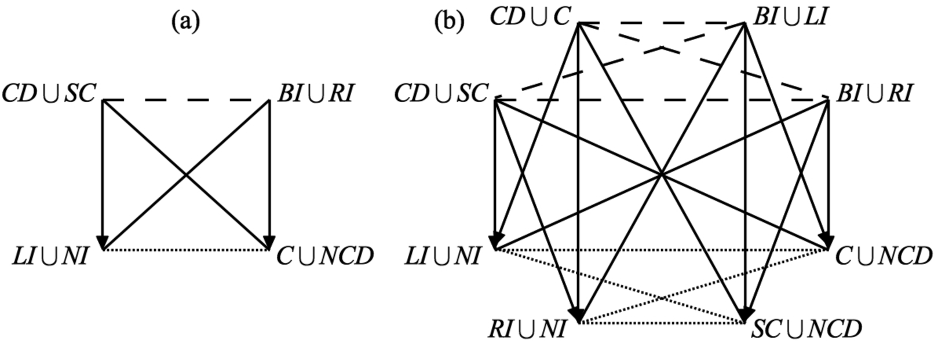
- note that these are *weak* opposition and implication relations: $SC^*_w, LI_w, C_w, RI^*_w$
- these four indeed form a square, but this square is
  - *classical* iff the relations' first argument (φ) is assumed to be satisfiable
  - *degenerated* otherwise (Béziau: "an X of opposition")


(a)
CD∪C
BI∪LI
RI∪NI
SC∪NCD
(b)
CD∪C
BI∪LI
RI∪NI
SC∪NCD

- Seuren (2014): 6 relations, forming a JSB hexagon
  ⇒ translate into opposition/implication terminology
  - a *JSB* hexagon iff the relations' first argument is satisfiable
  - a *U4* (= partially degenerated JSB) hexagon otherwise



- recall Löbner's relations:
  - 4 weak opposition/implication relations: $SC^*_w, LI_w, C_w, RI^*_w$
  - classical square iff $\varphi \neq \bot$
- completely analogously:
  - 4 other weak opposition/implication relations: $SC_w, LI^*_w, C^*_w, RI_w$
  - classical square iff $\varphi \neq \top$
- combination of these two squares:
  - all 8 weak opposition/implication relations together
  - minimal assumption: contingency of φ ($\varphi \neq \bot$ and $\varphi \neq \top$)
  - interesting if we also assume contingency of ψ
- importance of the resulting octagon:
  - metalogical analogue of an octagon for syllogistics with subject negation (Keynes, Johnson, Hacker, Reichenbach)
  - duality at metalogical level (Libert 2012)


(a)
CD∪SC
BI∪RI
LI∪NI
C∪NCD
(b)
CD∪C
BI∪LI
CD∪SC
BI∪RI
LI∪NI
C∪NCD
RI∪NI
SC∪NCD

KU LEUVEN

A Single Example

- consider the set of binary propositional connectives $\mathbb{B}(\mathsf{CPL}) \times \mathbb{B}(\mathsf{CPL}) \to \mathbb{B}(\mathsf{CPL})$

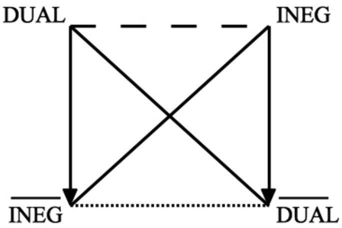
- claim: $\textsc{dual} \cap \textsc{ineg} = \emptyset$
  - if there exists $(O_1, O_2) \in \textsc{dual} \cap \textsc{ineg}$, then $O_1 = \neg O_2(\neg, \neg)$ and $O_1 = O_2(\neg, \neg)$, and hence $\neg O_2(\neg, \neg) = O_2(\neg, \neg)$, and hence $\neg O_2(\neg p, \neg q) \equiv_{\mathsf{CPL}} O_2(\neg p, \neg q)$, which is of the form $\neg\varphi \equiv_{\mathsf{CPL}} \varphi$ ↯

- claim: $\textsc{dual} \cup \textsc{ineg} \neq \mathbb{B}(\mathsf{CPL})^{\mathbb{B}(\mathsf{CPL}) \times \mathbb{B}(\mathsf{CPL})} \times \mathbb{B}(\mathsf{CPL})^{\mathbb{B}(\mathsf{CPL}) \times \mathbb{B}(\mathsf{CPL})}$
  - there are pairs of binary propositional connectives that are neither each other's duals nor each other's internal negations (e.g. $\wedge$ and $\to$)

- hence, $\textsc{dual}$ and $\textsc{ineg}$ are contraries

- this gives rise to an Aristotelian square

A Single Example



KU LEUVEN

- a logical diagram depends on two parameters:
  - decoration: the elements occurring in the diagram (vertices)
  - type: the type of logical relations between those elements (edges)

- in this talk:

type \ deco.	Aristotelian	opposition	implication	duality
Aristotelian	●	●	●	●
opposition	—	—	—	—
implication	—	—	—	—
duality	○	○	○	○

- construct Aristotelian (and other) diagrams with metalogical decorations (in a mathematically precise sense; not just "loosely speaking")
- various connections, observations and techniques:
  - connections between families of diagrams (JSB, SC, Buridan, RDH)
  - connections between authors (Béziau, Seuren, Löbner, Libert)
  - linguistic observations (strong/weak contrariety)
  - dependence on additional assumptions (satisfiability of 1st argument)
  - bitstring semantics (length 4 bitstrings for RDH)
- these are the counterparts of similar (and well-studied) connections, observations, techniques at the object-logical level

$\Rightarrow$ fundamental continuity between object- and metalogical decorations!

Thank you!

More info: www.logicalgeometry.org