Ordering relations, partitions and Aristotelian diagrams

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LNAT 4, Brussels
Structure of the talk

1. Introduction
2. The categorical statements from syllogistics
3. Propositional logic
4. Total ordering relations
5. Partial ordering relations
6. Total ordering relations, once again
7. Conclusion
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scalarity in mathematics: \textbf{ordering relations}

partial ordering $\leq$ on a set $D$:
- reflexivity: $\forall x \in D : x \leq x$
- transitivity: $\forall x, y, z \in D : x \leq y, y \leq z \Rightarrow x \leq z$
- antisymmetry: $\forall x, y \in D : x \leq y, y \leq x \Rightarrow x = y$

total ordering $\leq$ on a set $D$:
- all the properties of partial orderings
- totality: $\forall x, y \in D : x \leq y \text{ or } y \leq x$

today: the role of ordering relations in \textbf{logical geometry}
systematic study of the well-known **Aristotelian relations**: two statements are said to be

- **contradictory** iff they cannot be true together and they cannot be false together
- **contrary** iff they cannot be true together but they can be false together
- **subcontrary** iff they can be true together but they cannot be false together
- **in subalternation** iff the first proposition entails the second but the second doesn’t entail the first

**an Aristotelian diagram** is a visual representation of

- a fragment $\mathcal{F}$ of formulas (/natural language expressions/…)
- the Aristotelian relations holding between those formulas
consider a fragment of formulas $\mathcal{F}$

the **partition** of logical space that is induced by $\mathcal{F}$ is

$$\Pi(\mathcal{F}) := \{ \alpha \in \mathcal{L} \mid \alpha \equiv \pm \varphi_1 \land \cdots \land \pm \varphi_m, \text{ and } \alpha \text{ is consistent} \}$$

the elements of $\Pi(\mathcal{F})$ are called **anchor formulas**

ordering relations/scaliarty phenomena can play a role in the fragment $\mathcal{F}$ as well as in the partition $\Pi(\mathcal{F})$

**diagrammatic** representation:

```
logical realm   fragment $\mathcal{F}$   induces   partition $\Pi(\mathcal{F})$
↓           ↓                         ↓
visual realm  Aristotelian diagram  partition diagram
```
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consider the fragment of the four **categorical statements**:

\[ F_c := \{ \text{all humans are rational,} \]
\[ \text{some humans are rational,} \]
\[ \text{no humans are rational,} \]
\[ \text{some humans are not rational} \} \]

**note**: \( F_c \) does **not** seem to exhibit any **ordering** relation
The categorical statements from syllogistics

- fragment $F_c$ of the four categorical statements
- Aristotelian diagram for $F_c$: **classical square of opposition** (under the assumption of existential import)
The categorical statements from syllogistics

- fragment $\mathcal{F}_c$ of the four categorical statements
- the partition induced by $\mathcal{F}_c$:

$$\Pi(\mathcal{F}_c) = \{ \text{all humans are rational,}$$
$$\text{some but not all humans are rational,}$$
$$\text{no humans are rational}\}$$

- (the size of) the partition $\Pi(\mathcal{F}_c)$ allows us to measure the Boolean complexity of the fragment $\mathcal{F}_c$

- $|\Pi(\mathcal{F}_c)| = 3$
- the Boolean closure of $\mathcal{F}_c$ contains $2^3 = 8$ formulas
- up to logical equivalence, there are 8 Boolean combinations of $\mathcal{F}_c$-formulas
The categorical statements from syllogistics

- the partition induced by $\mathcal{F}_c$:

$$\Pi(\mathcal{F}_c) = \{ \text{all humans are rational,} \text{ some but not all humans are rational,} \text{ no humans are rational} \}$$

- diagrammatic representations of $\Pi(\mathcal{F}_c)$:

- note: $\Pi(\mathcal{F}_c)$ constitutes a total ordering of logical space
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• consider the fragment $\mathcal{F}_1$, which contains four formulas from **propositional logic**:

$$\mathcal{F}_1 := \{ p \land q, 
\quad p \lor q, 
\quad \neg p \land \neg q, 
\quad \neg p \lor \neg q \}$$

• note: $\mathcal{F}_1$ does **not** exhibit any **ordering** relation
• fragment $\mathcal{F}_1$ of four formulas from propositional logic

• Aristotelian diagram for $\mathcal{F}_1$: **classical square of opposition**
consider the fragment $\mathcal{F}_2$, which again contains four formulas from propositional logic:

$$\mathcal{F}_2 := \{ p, q, \neg p, \neg q \}$$

note: $\mathcal{F}_2$ does not exhibit any ordering relation
fragment $\mathcal{F}_2$ of four formulas from propositional logic

Aristotelian diagram for $\mathcal{F}_2$: **degenerate square of opposition**
- contradictions between $p/\neg p$ and $q/\neg q$
- all other pairs of formulas are unconnected: they do not stand in any Aristotelian relation at all
Propositional logic

- the partition induced by $\mathcal{F}_2$:

$$\Pi(\mathcal{F}_2) = \{ p \land q, \\
p \land \neg q, \\
\neg p \land q, \\
\neg p \land \neg q \}$$

- (the size of) the partition $\Pi(\mathcal{F}_2)$ allows us to measure the Boolean complexity of the fragment $\mathcal{F}_2$

  - $|\Pi(\mathcal{F}_2)| = 4$
  - the Boolean closure of $\mathcal{F}_2$ contains $2^4 = 16$ formulas
  - up to logical equivalence, there are 16 Boolean combinations of $\mathcal{F}_2$-formulas
diagrammatic representations of $\Pi(\mathcal{F}_2)$:

\[\begin{array}{cc}
p \land q & p \land \lnot q \\
\lnot p \land q & \lnot p \land \lnot q \\
\end{array}\]

- note: $\Pi(\mathcal{F}_2)$ does **not** involve any underlying **ordering** of logical space
- $\Pi(\mathcal{F}_2)$ displays a high degree of **symmetry**
- $\Pi(\mathcal{F}_2)$ is the result of crosscutting the two bipartitions $p/\lnot p$ and $q/\lnot q$
one might argue that $\Pi(\mathcal{F}_2)$ is an ordering of logical space after all:
- not a total ordering, but a partial ordering
- anchor formulas are ordered by ‘number of true (non-negated) conjuncts’

however, in most concrete cases, this does not seem very plausible
e.g. the crosscut of the bipartitions male/female and adult/child
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consider the fragment $\mathcal{F}_t$ of six statements involving a total ordering relation $\leq$ on a set $D$ and two elements $x, y \in D$:

$$\mathcal{F}_t := \{ \begin{array}{l} x > y, \\ x = y, \\ x < y, \\ x \leq y, \\ x \neq y, \\ x \geq y \end{array} \}$$
Aristotelian diagram for $\mathcal{F}_t$: a hexagon of opposition

- originally due to Robert Blanché (*Sur l’opposition des concepts*, 1953)
Total ordering relations

- the partition induced by $\mathcal{F}_t$:

$$\Pi(\mathcal{F}_t) = \{ x > y, \\
                x = y, \\
                x < y \}$$

- (the size of) the partition $\Pi(\mathcal{F}_t)$ allows us to measure the Boolean complexity of the fragment $\mathcal{F}_t$

  - $|\Pi(\mathcal{F}_t)| = 3$
  - the Boolean closure of $\mathcal{F}_t$ contains $2^3 = 8$ formulas
  - up to logical equivalence, there are 8 Boolean combinations of $\mathcal{F}_t$-formulas
  - apart from $\bot$ and $\top$, all of these Boolean combinations can already be found in the hexagon itself
  - the hexagon is closed under the Boolean operations
the partition induced by $\mathcal{F}_t$:

$$\Pi(\mathcal{F}_t) = \{ x > y, \quad x = y, \quad x < y \}$$

diagrammatic representations of $\Pi(\mathcal{F}_t)$:

- note: $\Pi(\mathcal{F}_t)$ constitutes itself a total ordering of logical space
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let $\mathcal{F}_p$ be exactly the same fragment as before ($\mathcal{F}_t$), but now under the assumption that $\leq$ is a partial ordering on $D$ instead of a total ordering

$\mathcal{F}_p := \{ x > y, 
\ x = y, 
\ x < y, 
\ x \leq y, 
\ x \neq y, 
\ x \geq y \}$

we drop the assumption of totality ($\forall x, y \in D : x \leq y$ or $y \leq x$)

it becomes possible for $x$ and $y$ to be incomparable: $x \not\approx y$
(i.e. neither $x \geq y$ nor $x \leq y$)
the Aristotelian diagram for $\mathcal{F}_p$:

a very different **hexagon of opposition**

- two of the three contradictions change into contrarieties ($> / \leq$ and $< / \geq$)
- one of the three subcontrarieties is lost ($\geq / \leq$)
- the three contrarieties and six subalternations remain unchanged
the partition induced by \( \mathcal{F}_p \):
\[
\Pi(\mathcal{F}_p) = \{ x > y, \quad x = y, \quad x < y, \quad x \neq y \}
\]

(the size of) the partition \( \Pi(\mathcal{F}_p) \) allows us to measure the Boolean complexity of the fragment \( \mathcal{F}_p \):

- \( |\Pi(\mathcal{F}_t)| = 4 \)
- the Boolean closure of \( \mathcal{F}_p \) contains \( 2^4 = 16 \) formulas
- up to logical equivalence, there are 16 Boolean combinations of \( \mathcal{F}_p \)-formulas
diagrammatic representations of $\Pi(\mathcal{F}_p)$:

<table>
<thead>
<tr>
<th>$x &gt; y$</th>
<th>$x = y$</th>
<th>$x &lt; y$</th>
</tr>
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<tbody>
<tr>
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</tbody>
</table>

$x \# y$

note: $\Pi(\mathcal{F}_p)$ constitutes itself a **partial ordering** of logical space
by setting $\#$ to be $\emptyset$
(i.e. imposing the requirement that $x \# y$ is impossible):
- from partial ordering to total ordering
- from the 4-partition $\Pi(\mathcal{F}_p)$ to 3-partition $\Pi(\mathcal{F}_t)$
- from Boolean closure of size $2^4 = 16$ to Boolean closure of size $2^3 = 8$

<table>
<thead>
<tr>
<th>partial ordering</th>
<th>total ordering</th>
<th>total ordering</th>
<th>partial ordering</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&gt;$</td>
<td>$\rightarrow$</td>
<td>$&gt;$</td>
<td>$= U &lt;$</td>
</tr>
<tr>
<td>$&gt; U #$</td>
<td>$\rightarrow$</td>
<td>$=$</td>
<td>$\leftarrow$ $= U &lt; U #$</td>
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<td>$&gt; U &lt;$</td>
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<tr>
<td>$#$</td>
<td>$\rightarrow$</td>
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<td>$&gt; U = U &lt;$</td>
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<td>$\emptyset$</td>
<td>$\rightarrow$</td>
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<td>$\leftarrow$ $&gt; U = U &lt;$</td>
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<td></td>
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<td>$&gt;$</td>
<td>$\leftarrow$ $&gt; U = U &lt; U #$</td>
</tr>
</tbody>
</table>
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7. Conclusion
Total ordering relations, once again

- so far:
  - focus on total ordering versus partial ordering
  - focus on the axiom of **totality**

- now:
  - focus on the axiom of **transitivity**
  - $\forall x, y, z \in D : x \leq y, y \leq z \Rightarrow x \leq z$

- consider the fragment $\mathcal{F}^*$, which, for three elements $x, y, z \in D$, contains all formulas of the form $x \circ y$, $y \circ z$ and $x \circ z$, with $\circ \in \{>, =, <, \leq, \neq, \geq\}$

- note: $|\mathcal{F}^*| = 3 \times 6 = 18$

- what is the partition $\Pi(\mathcal{F}^*)$ that is induced by $\mathcal{F}^*$?
Total ordering relations, once again

- we can write $F^* = F_{xy} \cup F_{yz} \cup F_{xz}$
  - $F_{xy} = \{x > y, x = y, x < y, x \leq y, x \neq y, x \geq y\}$
  - $F_{yz} = \{y > z, y = z, y < z, y \leq z, y \neq z, y \geq z\}$
  - $F_{xz} = \{x > z, x = z, x < z, x \leq z, x \neq z, x \geq z\}$
  - (Blanché hexagon) (Blanché hexagon) (Blanché hexagon)

- we know the partitions that are induced by these subfragments of $F^*$:
  - $\Pi(F_{xy}) = \{x > y, x = y, x < y\}$
  - $\Pi(F_{yz}) = \{y > z, y = z, y < z\}$
  - $\Pi(F_{xz}) = \{x > z, x = z, x < z\}$

- $\Pi(F^*)$ is the result of crosscutting $\Pi(F_{xy})$, $\Pi(F_{yz})$ and $\Pi(F_{xz})$
  - in principle $3 \times 3 \times 3 = 27$ conjunctions of anchor formulas
  - because of transitivity, many of these conjunctions are inconsistent (e.g. $x > y$, $y > z$, and $x < z$ are inconsistent with each other)
  - exactly 13 conjunctions are consistent, and thus get included in $\Pi(F^*)$
The partition $\Pi(\mathcal{F}^*)$ contains the following 13 formulas:

<table>
<thead>
<tr>
<th></th>
<th>Formula</th>
<th>Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$x &gt; y \land y &gt; z \land x &gt; z$</td>
<td>$x \mid y \mid z$</td>
</tr>
<tr>
<td>2</td>
<td>$x = y \land y &gt; z \land x &gt; z$</td>
<td>$xy \mid z$</td>
</tr>
<tr>
<td>3</td>
<td>$x &lt; y \land y &gt; z \land x &gt; z$</td>
<td>$y \mid x \mid z$</td>
</tr>
<tr>
<td>4</td>
<td>$x &gt; y \land y = z \land x &gt; z$</td>
<td>$x \mid yz$</td>
</tr>
<tr>
<td>5</td>
<td>$x &gt; y \land y &lt; z \land x &gt; z$</td>
<td>$x \mid z \mid y$</td>
</tr>
<tr>
<td>6</td>
<td>$x &lt; y \land y &gt; z \land x = z$</td>
<td>$y \mid xz$</td>
</tr>
<tr>
<td>7</td>
<td>$x = y \land y = z \land x = z$</td>
<td>$xyz$</td>
</tr>
<tr>
<td>8</td>
<td>$x &gt; y \land y &lt; z \land x = z$</td>
<td>$xz \mid y$</td>
</tr>
<tr>
<td>9</td>
<td>$x &lt; y \land y &gt; z \land x &lt; z$</td>
<td>$y \mid z \mid x$</td>
</tr>
<tr>
<td>10</td>
<td>$x &lt; y \land y = z \land x &lt; z$</td>
<td>$yz \mid x$</td>
</tr>
<tr>
<td>11</td>
<td>$x &gt; y \land y &lt; z \land x &lt; z$</td>
<td>$z \mid x \mid y$</td>
</tr>
<tr>
<td>12</td>
<td>$x = y \land y &lt; z \land x &lt; z$</td>
<td>$z \mid xy$</td>
</tr>
<tr>
<td>13</td>
<td>$x &lt; y \land y &lt; z \land x &lt; z$</td>
<td>$z \mid y \mid x$</td>
</tr>
</tbody>
</table>
(the size of) the partition $\Pi(\mathcal{F}^*)$ allows us to measure the Boolean complexity of the fragment $\mathcal{F}^*$. 

- recall that $|\mathcal{F}^*| = 18$
- we have just seen that $|\Pi(\mathcal{F}^*)| = 13$
- the Boolean closure of $\mathcal{F}^*$ contains $2^{13} = 8192$ formulas
- up to logical equivalence, there are 8192 Boolean combinations of $\mathcal{F}^*$-formulas

- the partition $\Pi(\mathcal{F}^*)$ is not an ordering on logical space, but rather has a high degree of symmetry

  - 6 conjunctions with 0 identity-conjuncts
  - 6 conjunctions with 1 identity-conjunct
  - 1 conjunction with 3 identity-conjuncts
Total ordering relations, once again

- a diagrammatic representation of $\Pi(\mathcal{F}^*)$
another diagrammatic representation of $\Pi(\mathcal{F}^*)$
(geometric combinatorics: permutahedron)
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ordering relations/scality phenomena can play a role
in the fragment $\mathcal{F}$ as well as in the partition $\Pi(\mathcal{F})$

<table>
<thead>
<tr>
<th>fragment / Aristotelian diagram</th>
<th>partition / partition diagram</th>
<th>concrete example</th>
</tr>
</thead>
<tbody>
<tr>
<td>not order-based</td>
<td>order-based</td>
<td>cf. section 2: $\mathcal{F}_c$</td>
</tr>
<tr>
<td>not order-based</td>
<td>not order-based</td>
<td>cf. section 3: $\mathcal{F}_2$</td>
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<tr>
<td>order-based</td>
<td>order-based</td>
<td>cf. sections 4,5: $\mathcal{F}_t, \mathcal{F}_p$</td>
</tr>
<tr>
<td>order-based</td>
<td>not order-based</td>
<td>cf. section 6: $\mathcal{F}^*$</td>
</tr>
</tbody>
</table>
Thank you!

Questions?

More info: www.logicalgeometry.org