Buridan’s and Avicenna’s Aristotelian Diagrams for Combined Operators

Lorenz Demey

CLAW/DWMC Symposium, 22 May ’15
Before we get started...

- this talk is based on joint research with
  - Saloua Chatti (U. Tunis)
  - Hans Smessaert (KU Leuven)
  - Fabien Schang (HSE Moscow)

- mix of logical and historical aspects
- today’s talk:
  - first half: emphasis on the historical aspects
  - second half: emphasis on the more technical aspects

- historical scholarship:
  - Buridan: S. Read, G. Hughes, S. Johnston, J. Campos Benítez
  - Avicenna: S. Chatti, W. Hodges

- status of diagrams:
  - heavyweight: visual representation of logical theory
  - lightweight: visual representation of logical theory
Goals of the talk

- Buridan’s Aristotelian octagons:
  - relatively well-known
  - actual diagrams

- logical goals:
  - systematically study some natural extensions of Buridan’s octagon
  - compare them in terms of their logical complexity (bitstring length)

- historical goals:
  - show that although he did not draw the actual diagram, Buridan had the logical means available to construct at least one of these extensions
  - establish the historical priority of Al-Farabi and Avicenna with respect to Buridan’s octagon and at least two of its extensions
1 Some Preliminaries from Logical Geometry

2 Buridan’s Aristotelian Diagrams

3 Avicenna’s Aristotelian Diagrams

4 Bitstring Analysis

5 Conclusion
Structure of the talk

1. Some Preliminaries from Logical Geometry
2. Buridan’s Aristotelian Diagrams
3. Avicenna’s Aristotelian Diagrams
4. Bitstring Analysis
5. Conclusion
an Aristotelian diagram visualizes some formulas and the Aristotelian relations holding between them

definition of the Aristotelian relations: two propositions are

contradictory  iff  they cannot be true together and they cannot be false together,

contrary  iff  they cannot be true together but they can be false together,

subcontrary  iff  they can be true together but they cannot be false together,

in subalternation  iff  the first proposition entails the second but the second doesn’t entail the first
Some Aristotelian squares

\( \text{all}(S, P) \quad \text{all}(S, \neg P) \)

\( \text{some}(S, P) \quad \text{some}(S, \neg P) \)

\( \Box p \quad \Box \neg p \)

\( \Diamond p \quad \Diamond \neg p \)

\( p \land q \quad \neg p \land \neg q \)

\( p \lor q \quad \neg p \lor \neg q \)

\( \langle \neg p \rangle q \quad \langle \neg p \rangle \neg q \)

\( \langle p \rangle q \quad \langle p \rangle \neg q \)
already during the Middle Ages, philosophers used Aristotelian diagrams larger than the classical square to visualize their logical theories

e.g. John Buridan (ca. 1295–1358): several octagons (see later)

e.g. William of Sherwood (ca. 1200–1272), *Introductiones in Logicam* ⇒ integrating singular propositions into the classical square
Boolean closure of an Aristotelian diagram

- the smallest Aristotelian diagram that contains all contingent Boolean combinations of formulas from the original diagram

- the Boolean closure of a classical square is a Jacoby-Sesmat-Blanché hexagon (6 formulas)
• the smallest Aristotelian diagram that contains all contingent Boolean combinations of formulas from the original diagram

• the Boolean closure of a classical square is a Jacoby-Sesmat-Blanché hexagon (6 formulas)

• the Boolean closure of a Sherwood-Czezowski hexagon is a (3D) rhombic dodecahedron (14 formulas)

Theorem

A Boolean closure has \(2^n - 2\) formulas, for some natural number \(n\).
every Aristotelian diagram can be represented by means of bitstrings

bitstring = sequence of bits (0/1)

‘anchor formulas’ \( \alpha_1, \ldots, \alpha_n \) (obtainable from the diagram)

every formula in (the Boolean closure of) the diagram is equivalent to a disjunction of these anchor formulas

bitstrings keep track which anchor formulas occur in the disjunction and which ones do not

technical: disjunctive normal forms

intuition: bitstrings as coordinates, anchor formulas as axes

\[
\text{point } (5, 2) \iff 5 \cdot \tilde{x} + 2 \cdot \tilde{y}
\]

bitstrings of length \( n \) \( \iff \) size of Boolean closure is \( 2^n - 2 \)
• **example:** modal square \( \Rightarrow \) bitstrings of length \( n = 3 \)

• **anchor formulas:**
  
  \[
  \alpha_1 = \Box p \\
  \alpha_2 = \diamond p \land \diamond \neg p \\
  \alpha_3 = \square \neg p
  \]

  e.g. \( \diamond p \equiv \Box p \lor (\diamond p \land \diamond \neg p) = \alpha_1 \lor \alpha_2 = 110 \)
• example: modal square ⇒ bitstrings of length \( n = 3 \)
• anchor formulas:
  \[ \alpha_1 = \Box p \]
  \[ \alpha_2 = \Diamond p \land \Diamond \neg p \]
  \[ \alpha_3 = \Box \neg p \]

\[ \Diamond p \equiv \Box p \lor (\Diamond p \land \Diamond \neg p) = \alpha_1 \lor \alpha_2 = 110 \]
Structure of the talk

1. Some Preliminaries from Logical Geometry
2. Buridan’s Aristotelian Diagrams
3. Avicenna’s Aristotelian Diagrams
4. Bitstring Analysis
5. Conclusion
Buridan’s Aristotelian diagrams

- John Buridan, ca. 1295–1358
- *Summulae de Dialectica* (late 1330s, revisions into the 1350s)
- Vatican manuscript Pal.Lat. 994 contains several Aristotelian diagrams:
  - Aristotelian square for the usual categorical propositions (A,I,E,O) (e.g. “every human is mortal”)
  - Aristotelian octagon for non-normal propositions (e.g. “every human some animal is not”) (cf. regimentation of Latin)
  - Aristotelian octagon for propositions with oblique terms (e.g. “every donkey of every human is running”)
  - Aristotelian octagon for modal propositions (e.g. “every human is necessarily mortal”)

\[
\begin{align*}
\text{square} & \Rightarrow \text{single operator} \\
\text{octagons} & \Rightarrow \text{combined operators}
\end{align*}
\]
Buridan’s square for the categorical propositions

Aristotelian Diagrams for Combined Operators – L. Demey
Buridan’s octagon for non-normal propositions

Aristotelian Diagrams for Combined Operators – L. Demey
Buridan’s octagon for propositions with oblique terms

Aristotelian Diagrams for Combined Operators – L. Demey
Buridan’s octagon for modal propositions
Buridan’s octagon contains the following 8 formulas:

1. all A are necessarily B
   \[ \forall x (\ Diamond A x \rightarrow \ Box B x) \]
2. all A are possibly B
   \[ \forall x (\ Diamond A x \rightarrow \ Diamond B x) \]
3. some A are necessarily B
   \[ \exists x (\ Diamond A x \land \ Box B x) \]
4. some A are possibly B
   \[ \exists x (\ Diamond A x \land \ Diamond B x) \]
5. all A are necessarily not B
   \[ \forall x (\ Diamond A x \rightarrow \ Box \neg B x) \]
6. all A are possibly not B
   \[ \forall x (\ Diamond A x \rightarrow \ Diamond \neg B x) \]
7. some A are necessarily not B
   \[ \exists x (\ Diamond A x \land \ Box \neg B x) \]
8. some A are possibly not B
   \[ \exists x (\ Diamond A x \land \ Diamond \neg B x) \]

- note: de re modality, ampliation of the subject in modal formulas
- watch out with negative formulas:
  no A are necessarily B
  = no A are (necessarily B)
  = all A are not (necessarily B)
  = all A are possibly not B
Buridan’s modal octagon
The internal structure of Buridan’s modal octagon

(assumption: $\Box \varphi \rightarrow \Diamond \varphi$)
The internal structure of Buridan’s modal octagon

(assumption: $\exists x \Box Ax$ — amplified version of existential import!)
The internal structure of Buridan’s modal octagon

(note: unconnectedness square in the middle of the octagon)
S. Chatti, 2015, Al-Farabi on Modal Oppositions

Al-Farabi: ca. 873–950 (± 400 years before Buridan)
- identified the 8 formulas of Buridan’s octagon
- identified some of the Aristotelian relations of the octagon
  (but all relations are deducible from the ones identified by Al-Farabi)

unlike Buridan, Al-Farabi does not seem to have visualized his logical theorizing by means of an actual diagram

unlike Buridan, Al-Farabi was not explicit about the issue of ampliation
Historical precursor: Al-Farabi
we can define a bitstring representation for Buridan’s modal octagon
this makes use of bitstrings of **length 6**

6 anchor formulas:

1. $\forall \Box$
2. $\forall \Diamond \land \exists \Diamond \land \exists \Diamond \neg$
3. $\forall \Diamond \land \forall \Diamond \neg$
4. $\exists \Diamond \land \exists \Diamond \neg$
5. $\forall \Diamond \land \exists \Diamond \neg \land \exists \Diamond$
6. $\forall \neg \Box$

note: this means that the Boolean closure of the octagon contains $2^6 - 2 = 62$ formulas
Bitstrings for Buridan’s modal octagon
• classical square (representable by bitstrings of length 3)  
  ⇒ natural extension: JSB hexagon, i.e. its Boolean closure \(6 = 2^3 - 2\)

• Buridan’s modal octagon (representable by bitstrings of length 6)  
  ⇒ its Boolean closure has \(2^6 - 2 = 62\) formulas ⇒ too large!  
  ⇒ other, more ‘reasonable’ extensions of the octagon?

• key idea:

Buridan’s octagon for quantified modal logic can be seen as arising out of the interaction of a quantifier square and a modality square  
instead of taking the Boolean closure of the entire octagon, we can take the Boolean closure of its ‘component squares’
The quantifier square contains $\forall, \forall\neg, \exists, \exists\neg$.

The modality square contains $\Box, \Box\neg, \Diamond, \Diamond\neg$.
Interaction of a quantifier square and a modality square

- **square × square ⇒ 4 × 4 = 16 formulas**

<table>
<thead>
<tr>
<th></th>
<th>□</th>
<th>□¬</th>
<th>♦</th>
<th>♦¬</th>
</tr>
</thead>
<tbody>
<tr>
<td>∀</td>
<td>∀□</td>
<td>∀□¬</td>
<td>∀♦</td>
<td>∀♦¬</td>
</tr>
<tr>
<td>∀¬</td>
<td>∀¬□</td>
<td>∀¬□¬</td>
<td>∀¬♦</td>
<td>∀¬♦¬</td>
</tr>
<tr>
<td>∃</td>
<td>∃♦</td>
<td>∃♦¬</td>
<td>∃□</td>
<td>∃□¬</td>
</tr>
<tr>
<td>∃¬</td>
<td>∃¬♦</td>
<td>∃¬♦¬</td>
<td>∃¬□</td>
<td>∃¬□¬</td>
</tr>
</tbody>
</table>

- **these 16 formulas are pairwise equivalent:**

\[
\forall \neg \square \equiv \forall \neg
\]

\[
\forall \neg □ \equiv \forall
\]

\[
\forall
\]

\[
\forall
\]

\[
\forall
\]

\[
\forall
\]

\[
\forall
\]

\[
\forall
\]

\[
\forall
\]

\[
\forall
\]

\[
\forall
\]

\[
\forall
\]

\[
\forall
\]

\[
\forall
\]

\[
\forall
\]

\[
\forall
\]
Interaction of a quantifier square and a modality square

<table>
<thead>
<tr>
<th></th>
<th>□</th>
<th>□¬</th>
<th>◊</th>
<th>◊¬</th>
</tr>
</thead>
<tbody>
<tr>
<td>∀</td>
<td>∀□</td>
<td>∀□¬</td>
<td>∀◊</td>
<td>∀◊¬</td>
</tr>
<tr>
<td>∀¬</td>
<td>∀¬□</td>
<td>∀¬□¬</td>
<td>∀◊</td>
<td>∀◊¬</td>
</tr>
<tr>
<td>∃</td>
<td>∃◊</td>
<td>∃◊¬</td>
<td>∃□</td>
<td>∃□¬</td>
</tr>
<tr>
<td>∃¬</td>
<td>∃¬◊</td>
<td>∃¬◊¬</td>
<td>∃□</td>
<td>∃□¬</td>
</tr>
</tbody>
</table>

- up to logical equivalence, we arrive at \( \frac{4 \times 4}{2} = 8 \) formulas
- these are exactly the formulas found in Buridan’s modal octagon
- octagon = square \( \times \) square
A first extension of Buridan’s octagon

- Buridan octagon = quantifier square × modality square
- take the Boolean closure of these components separately
- recall that the Boolean closure of a square is a JSB hexagon

- quantifier square × modality hexagon
- quantifier hexagon × modality square
- quantifier hexagon × modality hexagon

- we will start by considering the first of these:
  
  quantifier square × modality hexagon
### A first extension of Buridan’s octagon

![Diagram of Buridan's octagon with four operators □, ◊, ∀, and ⊸, and their logical relationships.]

<table>
<thead>
<tr>
<th></th>
<th>□</th>
<th>¬□</th>
<th>◊</th>
<th>¬◊</th>
<th>□ ∨ □</th>
<th>¬□ ∨ ¬□</th>
<th>□ ∧ □</th>
<th>¬□ ∧ ¬□</th>
</tr>
</thead>
<tbody>
<tr>
<td>□</td>
<td>□</td>
<td>□</td>
<td>◊</td>
<td>◊</td>
<td>□ ∨ □</td>
<td>□ ∨ □</td>
<td>□ ∧ □</td>
<td>□ ∧ □</td>
</tr>
<tr>
<td>¬□</td>
<td>¬□</td>
<td>¬□</td>
<td>¬◊</td>
<td>¬◊</td>
<td>¬□ ∨ ¬□</td>
<td>¬□ ∨ ¬□</td>
<td>¬□ ∧ ¬□</td>
<td>¬□ ∧ ¬□</td>
</tr>
<tr>
<td>∀</td>
<td>∀</td>
<td>∀</td>
<td>∀</td>
<td>∀</td>
<td>∀(□ ∨ □)</td>
<td>∀(□ ∨ □)</td>
<td>∀(□ ∧ □)</td>
<td>∀(□ ∧ □)</td>
</tr>
<tr>
<td>¬∀</td>
<td>¬∀</td>
<td>¬∀</td>
<td>¬∀</td>
<td>¬∀</td>
<td>¬∀(□ ∨ □)</td>
<td>¬∀(□ ∨ □)</td>
<td>¬∀(□ ∧ □)</td>
<td>¬∀(□ ∧ □)</td>
</tr>
<tr>
<td>∃</td>
<td>∃</td>
<td>∃</td>
<td>∃</td>
<td>∃</td>
<td>∃(□ ∨ □)</td>
<td>∃(□ ∨ □)</td>
<td>∃(□ ∧ □)</td>
<td>∃(□ ∧ □)</td>
</tr>
<tr>
<td>¬∃</td>
<td>¬∃</td>
<td>¬∃</td>
<td>¬∃</td>
<td>¬∃</td>
<td>¬∃(□ ∨ □)</td>
<td>¬∃(□ ∨ □)</td>
<td>¬∃(□ ∧ □)</td>
<td>¬∃(□ ∧ □)</td>
</tr>
</tbody>
</table>

#### Aristotelian Diagrams for Combined Operators – L. Demey
A first extension of Buridan’s octagon

<table>
<thead>
<tr>
<th></th>
<th>□</th>
<th>□¬</th>
<th>◊</th>
<th>◊¬</th>
<th>□ ∨ □¬</th>
<th>◊ ∧ ◊¬</th>
</tr>
</thead>
<tbody>
<tr>
<td>∀</td>
<td>∀□</td>
<td>∀□¬</td>
<td>∀◊</td>
<td>∀◊¬</td>
<td>∀(□ ∨ □¬)</td>
<td>∀(◊ ∧ ◊¬)</td>
</tr>
<tr>
<td>∀¬</td>
<td>∀□¬</td>
<td>∀□</td>
<td>∀◊</td>
<td>∀◊¬</td>
<td>∀¬(□ ∨ □¬)</td>
<td>∀¬(◊ ∧ ◊¬)</td>
</tr>
<tr>
<td>∃</td>
<td>∃◊</td>
<td>∃◊¬</td>
<td>∃□</td>
<td>∃□¬</td>
<td>∃(□ ∨ □¬)</td>
<td>∃(◊ ∧ ◊¬)</td>
</tr>
<tr>
<td>∃¬</td>
<td>∃¬□</td>
<td>∃¬□¬</td>
<td>∃¬◊</td>
<td>∃¬◊¬</td>
<td>∃¬(□ ∨ □¬)</td>
<td>∃¬(◊ ∧ ◊¬)</td>
</tr>
</tbody>
</table>

- note: ∀(□ ∨ □¬) should be read as: ∀x(◊Ax → (□Bx ∨ □¬Bx))
- 8 new formulas, but again pairwise equivalent:
  - ∀∀¬(□ ∨ □¬) ≡ ∀(◊ ∧ ◊¬)  
  - ∃∀¬(□ ∨ □¬) ≡ ∃(◊ ∧ ◊¬)  
  - ∀∀¬(◊ ∧ ◊¬) ≡ ∀(□ ∨ □¬)  
  - ∃∀¬(◊ ∧ ◊¬) ≡ ∃(□ ∨ □¬)
A first extension of Buridan’s octagon

<table>
<thead>
<tr>
<th></th>
<th>□</th>
<th>□¬</th>
<th>◊</th>
<th>◊¬</th>
<th>□ ∨ □¬</th>
<th>◊ ∧ ◊¬</th>
</tr>
</thead>
<tbody>
<tr>
<td>∀</td>
<td>∀□</td>
<td>∀□¬</td>
<td>∀◊</td>
<td>∀◊¬</td>
<td>∀(□ ∨ □¬)</td>
<td>∀(◊ ∧ ◊¬)</td>
</tr>
<tr>
<td>∀¬</td>
<td>∀¬□</td>
<td>∀¬□¬</td>
<td>∀¬◊</td>
<td>∀¬◊¬</td>
<td>∀¬(□ ∨ □¬)</td>
<td>∀¬(◊ ∧ ◊¬)</td>
</tr>
<tr>
<td>∃</td>
<td>∃◊</td>
<td>∃◊¬</td>
<td>∃□</td>
<td>∃□¬</td>
<td>∃(□ ∨ □¬)</td>
<td>∃(◊ ∧ ◊¬)</td>
</tr>
<tr>
<td>∃¬</td>
<td>∃¬◊</td>
<td>∃¬◊¬</td>
<td>∃¬□</td>
<td>∃¬□¬</td>
<td>∃¬(□ ∨ □¬)</td>
<td>∃¬(◊ ∧ ◊¬)</td>
</tr>
</tbody>
</table>

- **Note:** ∀(□ ∨ □¬) should be read as: ∀x(◊Ax → (□Bx ∨ □¬Bx))
- **8 new formulas, but again pairwise equivalent:**
  - ∀¬(□ ∨ □¬) ≡ ∀(◊ ∧ ◊¬)
  - ∀¬(◊ ∧ ◊¬) ≡ ∀(□ ∨ □¬)
  - ∃¬(□ ∨ □¬) ≡ ∃(◊ ∧ ◊¬)
  - ∃¬(◊ ∧ ◊¬) ≡ ∃(□ ∨ □¬)

- Up to logical equivalence, we arrive at $\frac{4 \times 6}{2} = 12$ formulas
  ⇒ Aristotelian dodecagon that extends Buridan’s octagon

- More reasonable than the octagon’s full Boolean closure ($8 < 12 \ll 62$)
A first extension of Buridan’s octagon
Buridan’s works
- contain the octagon
- do not contain the dodecagon

S. Read, 2015, John Buridan on Non-Contingency Syllogisms
- identified the 12 formulas of the dodecagon
- identified the Aristotelian relations of the dodecagon

note: $\forall(\Box \lor \Box \neg) \neq \forall \Box \lor \forall \Box \neg$

Buridan: “this is true, ‘No planet is contingently the moon’, but this is false, ‘Every planet is necessarily the moon or every planet necessarily fails to be the moon’.” (Tractatus de Consequentiis)

$\text{not — contingently} = \forall \neg (\Diamond \land \Diamond \neg) \equiv \forall (\Box \lor \Box \neg) \neq \forall \Box \lor \forall \Box \neg$
Structure of the talk

1. Some Preliminaries from Logical Geometry
2. Buridan’s Aristotelian Diagrams
3. Avicenna’s Aristotelian Diagrams
4. Bitstring Analysis
5. Conclusion
Buridan “had” a dodecagon (quantifier square $\times$ modality hexagon)

S. Chatti, 2015, Les Carrés d’Avicenne

Avicenna: ca. 980–1037 ($\pm$ 300 years before Buridan)
- identified the 12 formulas of the dodecagon
- identified the Aristotelian relations of the dodecagon

but with temporal instead of modal operators

<table>
<thead>
<tr>
<th>formula</th>
<th>Buridan</th>
<th>Avicenna</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\exists \Box$</td>
<td>some A are <strong>necessarily</strong> B</td>
<td>some A are <strong>always</strong> B</td>
</tr>
<tr>
<td>$\forall \Diamond$</td>
<td>all A are <strong>possibly</strong> B</td>
<td>all A are <strong>sometimes</strong> B</td>
</tr>
</tbody>
</table>

Buridan: dodecagon $=$ quantifier square $\times$ modal hexagon
Avicenna: dodecagon $=$ quantifier square $\times$ temporal hexagon
the story so far:

- Buridan: octagon = quantifier square $\times$ modality square
- first extension: take Boolean closure of the second square
  $\Rightarrow$ dodecagon = quantifier square $\times$ modality hexagon

now: second extension: take Boolean closure of the first square
  $\Rightarrow$ dodecagon = quantifier hexagon $\times$ modality square

but also switch the roles of quantifiers and modalities
  $\Rightarrow$ dodecagon = modality hexagon $\times$ quantifier square

(from de re modalities to de dicto modalities)
A second extension of Buridan’s octagon

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>¬A</th>
<th>E</th>
<th>¬E</th>
</tr>
</thead>
<tbody>
<tr>
<td>□</td>
<td>□</td>
<td>□¬A</td>
<td>E</td>
<td>¬E</td>
</tr>
<tr>
<td>□</td>
<td>□</td>
<td>□¬A</td>
<td>E</td>
<td>¬E</td>
</tr>
<tr>
<td>◊</td>
<td>◊</td>
<td>▽A</td>
<td>E</td>
<td>¬E</td>
</tr>
<tr>
<td>◊</td>
<td>◊</td>
<td>▽A</td>
<td>E</td>
<td>¬E</td>
</tr>
<tr>
<td>◊</td>
<td>◊</td>
<td>▽A</td>
<td>E</td>
<td>¬E</td>
</tr>
<tr>
<td>◊</td>
<td>◊</td>
<td>▽A</td>
<td>E</td>
<td>¬E</td>
</tr>
<tr>
<td>□ ∧ □</td>
<td>□ ∧ □</td>
<td>□ ∧ □</td>
<td>E</td>
<td>¬E</td>
</tr>
<tr>
<td>□ ∧ □</td>
<td>□ ∧ □</td>
<td>□ ∧ □</td>
<td>E</td>
<td>¬E</td>
</tr>
</tbody>
</table>

Aristotelian Diagrams for Combined Operators – L. Demey
A second extension of Buridan’s octagon

<table>
<thead>
<tr>
<th></th>
<th>∀</th>
<th>¬∀</th>
<th>∃</th>
<th>¬∃</th>
<th>♦</th>
<th>¬♦</th>
</tr>
</thead>
<tbody>
<tr>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
</tr>
<tr>
<td>¬□</td>
<td>♦</td>
<td>♦</td>
<td>♦</td>
<td>♦</td>
<td>♦</td>
<td>♦</td>
</tr>
<tr>
<td>♦</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
</tr>
<tr>
<td>¬♦</td>
<td>♦</td>
<td>♦</td>
<td>♦</td>
<td>♦</td>
<td>♦</td>
<td>♦</td>
</tr>
<tr>
<td>□ ∨ □</td>
<td>∀</td>
<td>∀</td>
<td>∀</td>
<td>∀</td>
<td>∀</td>
<td>∀</td>
</tr>
<tr>
<td>¬□ ∨ ¬□</td>
<td>∀</td>
<td>∀</td>
<td>∀</td>
<td>∀</td>
<td>∀</td>
<td>∀</td>
</tr>
<tr>
<td>¬□ ∧ ¬□</td>
<td>∀</td>
<td>∀</td>
<td>∀</td>
<td>∀</td>
<td>∀</td>
<td>∀</td>
</tr>
<tr>
<td>□ ∨ □</td>
<td>∀</td>
<td>∀</td>
<td>∀</td>
<td>∀</td>
<td>∀</td>
<td>∀</td>
</tr>
</tbody>
</table>

- note: (□ ∨ □)∀ should be read as: □∀ ∨ □∀ (≡ □∀ ∨ □¬∀)
- 6 × 4 = 24 formulas, but again pairwise equivalent
A second extension of Buridan’s octagon

<table>
<thead>
<tr>
<th></th>
<th>∀</th>
<th>¬∀</th>
<th>∃</th>
<th>¬∃</th>
</tr>
</thead>
<tbody>
<tr>
<td>□</td>
<td>□∀</td>
<td>□¬∀</td>
<td>□∃</td>
<td>□¬∃</td>
</tr>
<tr>
<td>□¬</td>
<td>□∀</td>
<td>□¬∀</td>
<td>□¬∃</td>
<td>□¬¬∃</td>
</tr>
<tr>
<td>♦</td>
<td>♦∀</td>
<td>♦¬∀</td>
<td>♦∃</td>
<td>♦¬∃</td>
</tr>
<tr>
<td>□¬</td>
<td>□¬∀</td>
<td>□¬¬∀</td>
<td>□¬¬∃</td>
<td>□¬¬¬∃</td>
</tr>
<tr>
<td>□ ∨ □¬</td>
<td>□ ∨ □¬∀</td>
<td>□ ∨ □¬¬∀</td>
<td>□ ∨ □¬¬¬∀</td>
<td>□ ∨ □¬¬¬¬∀</td>
</tr>
<tr>
<td>□ ∧ □¬</td>
<td>□ ∧ □¬∀</td>
<td>□ ∧ □¬¬∀</td>
<td>□ ∧ □¬¬¬∀</td>
<td>□ ∧ □¬¬¬¬∀</td>
</tr>
</tbody>
</table>

**note:** (□ ∨ □¬∀) should be read as: □∀ ∨ □¬∀ (≡ □∀ ∨ □¬¬∀)

6 × 4 = 24 formulas, but again pairwise equivalent

up to logical equivalence, we arrive at \( \frac{6 \times 4}{2} = 12 \) formulas
⇒ another Aristotelian dodecagon that extends Buridan’s octagon
A second extension of Buridan’s octagon
Historical precursor: Avicenna

- S. Chatti, 2014, Avicenna on Possibility and Necessity

- Avicenna:
  - identified the 12 formulas of this second dodecagon
  - identified the Aristotelian relations holding between them
Structure of the talk

1. Some Preliminaries from Logical Geometry
2. Buridan’s Aristotelian Diagrams
3. Avicenna’s Aristotelian Diagrams
4. Bitstring Analysis
5. Conclusion
Bitstring analysis

- recall: Buridan octagon $\Rightarrow$ bitstrings of length 6

- anchor formulas:
  1. $\forall \Box$
  2. $\forall \Diamond \land \exists \Box \land \exists \Diamond$
  3. $\forall \Diamond \land \forall \Diamond$
  4. $\exists \Diamond \land \exists \Box$
  5. $\forall \Diamond \land \exists \Box \land \exists \Diamond$
  6. $\forall \Box$

- second extension (hexagon $\times$ square) $\Rightarrow$ bitstrings of length 6

- anchor formulas: same as above
  (except that quantifiers and modalities should be switched)

- this shows that the second extension of Buridan’s octagon remains within the latter’s Boolean closure
Bitstrings for the second extension

Aristotelian Diagrams for Combined Operators – L. Demey
first extension (square × hexagon) ⇒ bitstrings of length 7

anchor formulas:

1. \( \forall \square \)
2. \( \forall \diamond \land \exists \square \land \exists \diamond \neg \)
3. \( \forall \diamond \land \forall \diamond \neg \)
4a. \( \exists \square \land \exists \square \neg \land \exists (\diamond \land \diamond \neg) \)
4b. \( \exists \square \land \exists \square \neg \land \forall (\square \lor \square \neg) \)
5. \( \forall \diamond \neg \land \exists \square \neg \land \exists \diamond \)
6. \( \forall \square \neg \)

same as for the octagon, except that 4 has been ‘split’ into 4a and 4b

the first extension is essentially more complex than the original octagon

- the first extension does not fit within the octagon’s Boolean closure
- Boolean closure of the octagon: \( 2^6 - 2 = 62 \) formulas
- Boolean closure of the first extension: \( 2^7 - 2 = 126 \) formulas

why so many additional formulas?

- formulas where the quantifier does not distribute over the modality
- cf. anchor formulas 4a and 4b
Bitstrings for the first extension

Aristotelian Diagrams for Combined Operators – L. Demey
Structure of the talk

1. Some Preliminaries from Logical Geometry
2. Buridan’s Aristotelian Diagrams
3. Avicenna’s Aristotelian Diagrams
4. Bitstring Analysis
5. Conclusion
natural extension from a technical (and historical?) perspective:

- take Boolean closure of both square components
- so we get hexagon $\times$ hexagon $\Rightarrow \frac{6\times6}{2} = 18$ formulas
- e.g. “some but not all men are contingently philosophers”

overview:

<table>
<thead>
<tr>
<th>Buridan</th>
<th>8-gon</th>
<th>quantifier square</th>
<th>$\times$</th>
<th>modality square</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>“Al-Farabi”</td>
<td>8-gon</td>
<td>quantifier square</td>
<td>$\times$</td>
<td>modality square</td>
<td>6</td>
</tr>
<tr>
<td>“Buridan”</td>
<td>12-gon</td>
<td>quantifier square</td>
<td>$\times$</td>
<td>modality hexagon</td>
<td>7</td>
</tr>
<tr>
<td>“Avicenna”</td>
<td>12-gon</td>
<td>quantifier square</td>
<td>$\times$</td>
<td>temporal hexagon</td>
<td>7</td>
</tr>
<tr>
<td>“Avicenna”</td>
<td>12-gon</td>
<td>modality hexagon</td>
<td>$\times$</td>
<td>quantifier square</td>
<td>6</td>
</tr>
<tr>
<td>???</td>
<td>18-gon</td>
<td>quantifier hexagon</td>
<td>$\times$</td>
<td>modal hexagon</td>
<td>7</td>
</tr>
</tbody>
</table>
Thank you!

More info: www.logicalgeometry.org