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The Logical Geometry of Russell's Theory of Definite Descriptions

Lorenz Demey

Research Colloquium on Logic and Epistemology
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Purpurea
Iliace
Contrarie
Subalterne
Subalterne
Subcontrarie
Eventuli
Incipit Secundus Tractatus

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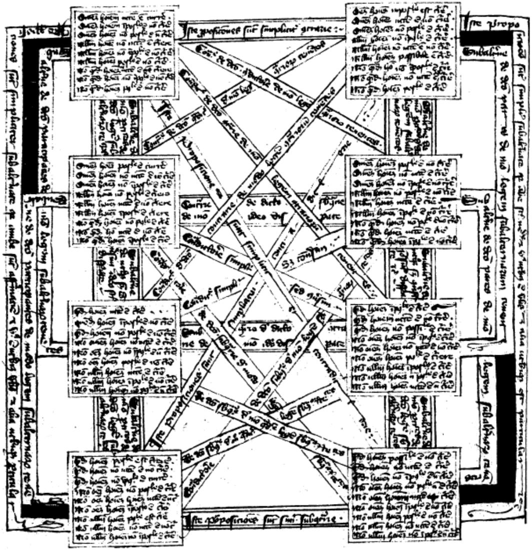
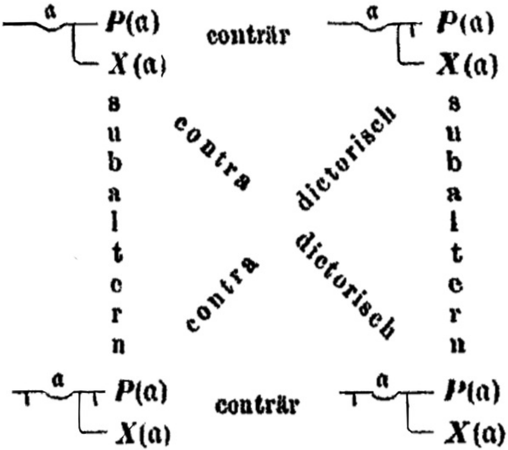



P(a)
X(a)
conträr
P(a)
X(a)
subaltern
contra
dictorisch
subaltern
contra
dictorisch
P(a)
X(a)
conträr
P(a)
X(a)

A Formal Concept View of Abstract Argumentation

Leila Amgoud and Henri Prade

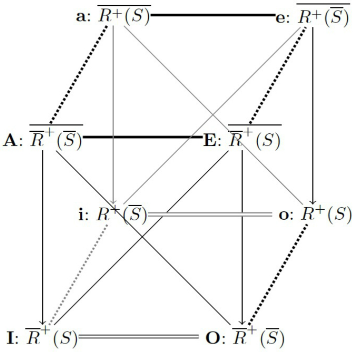
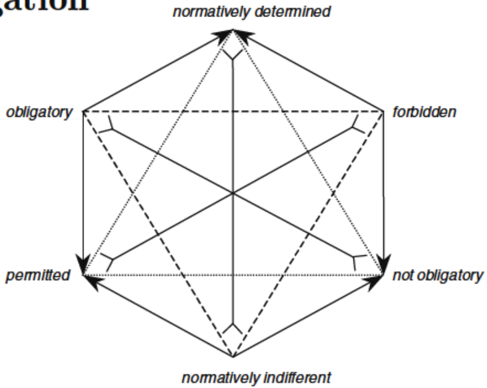


Fig. 1. Cube of oppositions between 8 remarkable sets of arguments

Deontological Square, Hexagon, and Decagon: A Deontic Framework for Supererogation

Jan C. Joerden

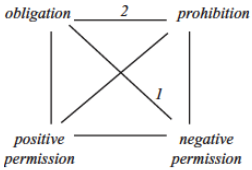


The European Journal of International Law Vol. 17 no.2

The Definition of 'Norm Conflict' in International Law and Legal Theory

Erich Vranes*

The possible set of inter-relations can be illustrated by using the so-called deontic square, which in fact relies on the logic square known since Greek antiquity,[85] and which was arguably first used in deontic logic by Bentham:[86]



Universal vs. particular reasoning: a study with neuroimaging techniques

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CLAUDIA CASADIO†, *Dipartimento di Filosofia, Università di Chieti-Pescara, Via dei Vestini, 66100 Chieti, Italy*

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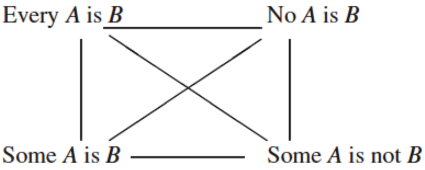
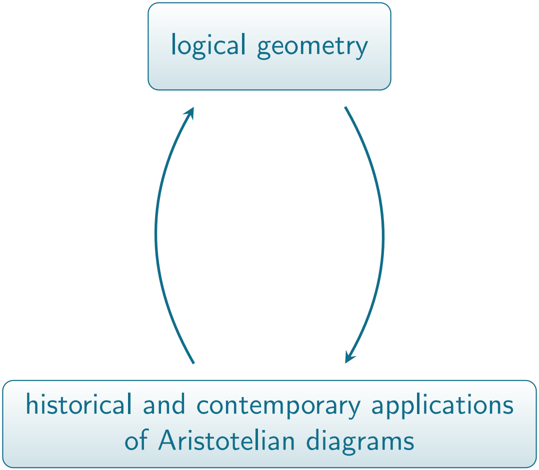
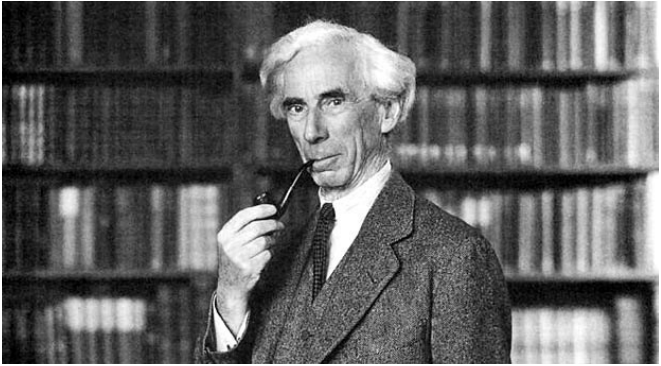



logical geometry
historical and contemporary applications
of Aristotelian diagrams



"ever since the beginning of the seventeenth century, almost every serious intellectual advance has had to begin with an attack on some Aristotelian doctrine; in logic, this is still true at the present day"

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- definite descriptions in natural language:
  - the president of the United States
  - the man standing over there
  - the so-and-so

- they can occur in
  - **subject position** e.g. The president was in Hamburg last week.
  - predicate position e.g. Donald Trump is currently still the president.

- Russell's quantificational analysis of 'the A is B'

$$\exists x\Big(Ax \wedge \forall y(Ay \rightarrow y = x) \wedge Bx\Big)$$

- Neale's restricted quantifier notation

$$[\text{the } x \colon Ax]Bx$$

- [the $x\colon Ax$]Bx $\equiv_{\mathsf{FOL}}$ (EX) $\wedge$ (UN) $\wedge$ (UV)

  (EX) $\exists x Ax$ — there exists at least one A
  (UN) $\forall x \forall y((Ax \wedge Ay) \rightarrow x = y)$ — there exists at most one A
  (UV) $\forall x(Ax \rightarrow Bx)$ — all As are B

- much of the subsequent literature on Russell's quantificational theory of definite descriptions has focused on one of these three conditions

- for a given logical system S (with Boolean connectives $\wedge, \neg$ and a model-theoretical semantics $\models$), the formulas $\varphi, \psi \in \mathcal{L}_\mathsf{S}$ are

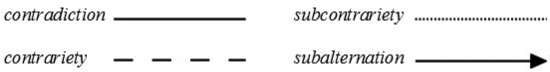
S-contradictory	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \models \neg(\neg\varphi \wedge \neg\psi)$
S-contrary	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \not\models \neg(\neg\varphi \wedge \neg\psi)$
S-subcontrary	iff	$\mathsf{S} \not\models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \models \neg(\neg\varphi \wedge \neg\psi)$
in S-subalternation	iff	$\mathsf{S} \models \varphi \to \psi$	and	$\mathsf{S} \not\models \psi \to \varphi$

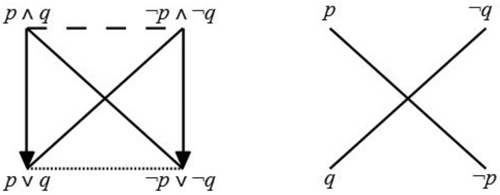
- 'φ and ψ cannot be true together'
  $\Rightarrow$ there exists no S-model $\mathbb{M}$ such that $\mathbb{M} \models \varphi \wedge \psi$
  $\Rightarrow$ for all S-models $\mathbb{M}$ it holds that $\mathbb{M} \models \neg(\varphi \wedge \psi)$
  $\Rightarrow$ $\mathsf{S} \models \neg(\varphi \wedge \psi)$

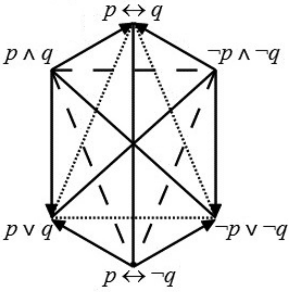
- 'φ and ψ can be false together'
  $\Rightarrow$ there exists a S-model $\mathbb{M}$ such that $\mathbb{M} \models \neg\varphi \wedge \neg\psi$
  $\Rightarrow$ $\mathsf{S} \not\models \neg(\neg\varphi \wedge \neg\psi)$

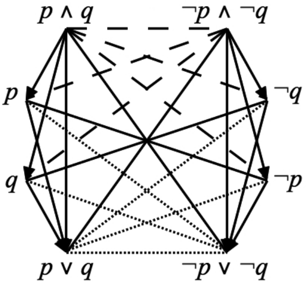
Aristotelian Diagrams

- Aristotelian diagram visualizes:
  - a (finite) set of S-contingent formulas
  - the Aristotelian relations holding among those formulas (in S)
- some basic examples from CPL (classical propositional logic):
  - classical square
  - degenerate square
  - Jacoby-Sesmat-Blanché (JSB) hexagon
  - Buridan octagon
- visual code:

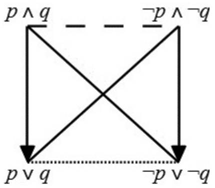


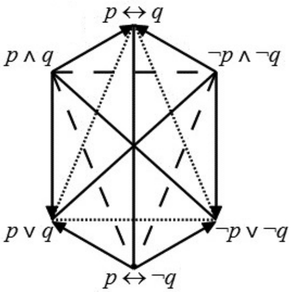

$p \wedge q$
$\neg p \wedge \neg q$
$p \vee q$
$\neg p \vee \neg q$
p
$\neg q$
q
$\neg p$


p ↔ q
p ∧ q
¬p ∧ ¬q
p ∨ q
¬p ∨ ¬q
p ↔ ¬q


p ∧ q
¬p ∧ ¬q
p
¬q
q
¬p
p ∨ q
¬p ∨ ¬q

- a diagram is *Boolean closed* iff it contains every contingent Boolean combination of its formulas (up to logical equivalence)
- *Boolean closure* of a diagram $D =$
  smallest Boolean closed diagram that contains D as a subdiagram





- for a given logic S and fragment $\mathcal{F}$ of formulas,
  define the partition $\Pi_{\mathsf{S}}(\mathcal{F}) := \{\bigwedge_{\varphi \in \mathcal{F}} \pm\varphi\} - \{\bot\}$
  - mutually exclusive: $\mathsf{S} \models \neg(\alpha_i \wedge \alpha_j)$ for distinct $\alpha_i, \alpha_j \in \Pi_{\mathsf{S}}(\mathcal{F})$
  - jointly exhaustive: $\mathsf{S} \models \bigvee \Pi_{\mathsf{S}}(\mathcal{F})$

- every $\varphi \in \mathcal{F}$ is S-equivalent to a disjunction of $\Pi_{\mathsf{S}}(\mathcal{F})$-formulas:
  $\varphi \equiv_{\mathsf{S}} \bigvee\{\alpha \in \Pi_{\mathsf{S}}(\mathcal{F}) \mid \mathsf{S} \models \alpha \rightarrow \varphi\}$ (relativized disjunctive normal form)

- bitstrings keep track which formulas enter into this disjunction
  - suppose $\Pi_{\mathsf{S}}(\mathcal{F}) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$
  - if $\varphi \equiv_{\mathsf{S}} \alpha_2 \vee \alpha_3 \vee \alpha_5$, then represent φ as the bitstring 01101

- bitstrings measure the Boolean complexity of $\mathcal{F}$
  - bitstring length: $|\Pi_{\mathsf{S}}(\mathcal{F})|$
  - the Boolean closure of $\mathcal{F}$ contains $2^{|\Pi_{\mathsf{S}}(\mathcal{F})|} - 2$ contingent formulas

- Russell: what is the negation of 'the A is B'?
  - law of excluded middle $\Rightarrow$ 'the A is B' is true or 'the A is not B' is true
  - but if there are no As, then both statements seem to be false
- Russell: 'the A is not B' is ambiguous (scope)
  - $\neg\exists x\Big(Ax \wedge \forall y(Ay \rightarrow y = x) \wedge Bx\Big)$ $\neg$[the x: $Ax]Bx$
  - $\exists x\Big(Ax \wedge \forall y(Ay \rightarrow y = x) \wedge \neg Bx\Big)$ [the x: $Ax]\neg Bx$
- first interpretation:
  - Boolean negation of 'the A is B'
  - if there are no As, then [the x: $Ax]Bx$ is false, $\neg$[the x: $Ax]Bx$ is true
- second interpretation:
  - if there are no As, then [the x: $Ax]Bx$ and [the x: $Ax]\neg Bx$ are false
  - not the Boolean negation of 'the A is B'

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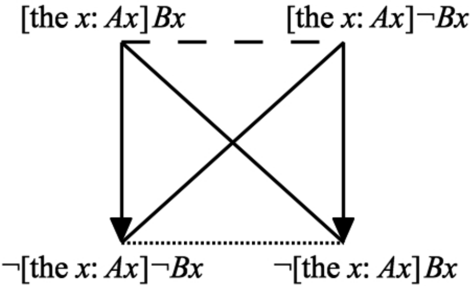
- crucial insight: the two interpretations of 'the A is not B' distinguished by Russell stand in different Aristotelian relations to 'the A is B'
  - $[\text{the } x\colon Ax]Bx$ and $\neg[\text{the } x\colon Ax]Bx$ are FOL-contradictory
  - $[\text{the } x\colon Ax]Bx$ and $[\text{the } x\colon Ax]\neg Bx$ are FOL-contrary

- cf. Haack (1965), Speranza and Horn (2010, 2012), Martin (2016)

- natural move: consider a fourth formula (with two negations)

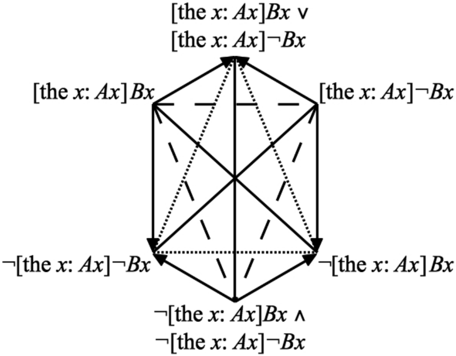
$$\begin{array}{rr} \exists x\big(Ax \wedge \forall y(Ay \to y = x) \wedge Bx\big) & [\text{the } x\colon Ax]Bx \\ \neg\exists x\big(Ax \wedge \forall y(Ay \to y = x) \wedge Bx\big) & \neg[\text{the } x\colon Ax]Bx \\ \exists x\big(Ax \wedge \forall y(Ay \to y = x) \wedge \neg Bx\big) & [\text{the } x\colon Ax]\neg Bx \\ \neg\exists x\big(Ax \wedge \forall y(Ay \to y = x) \wedge \neg Bx\big) & \neg[\text{the } x\colon Ax]\neg Bx \end{array}$$

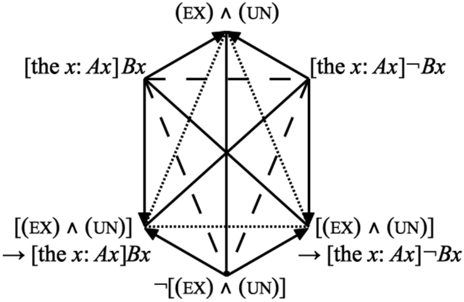
- in FOL, these four formulas constitute a classical square of opposition


[the x: Ax] Bx
[the x: Ax]¬Bx
¬[the x: Ax]¬Bx
¬[the x: Ax] Bx

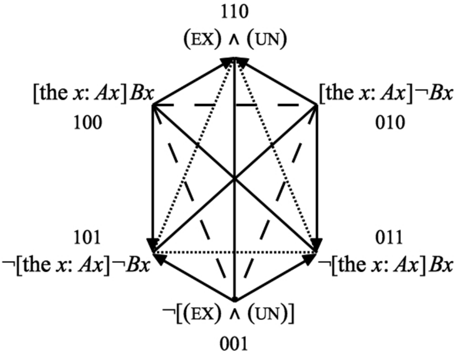
- this square is fully defined in ‘ordinary’ FOL $\Rightarrow$ acceptable for Russell
- summarizes Russell’s solution to puzzle on law of excluded middle
- interesting new formula: $\neg[\text{the } x\colon Ax]\neg Bx$
  - expresses a weak version of ‘the A is B’
    $\neg[\text{the } x\colon Ax]\neg Bx \quad \equiv_{\text{FOL}} \quad [(\textsc{ex}) \wedge (\textsc{un})] \rightarrow [\text{the } x\colon Ax]Bx$
    - if there is exactly one A,
      $[\text{the } x\colon Ax]Bx$ and $\neg[\text{the } x\colon Ax]\neg Bx$ always have the same truth value
    - in all other cases,
      $[\text{the } x\colon Ax]Bx$ is always false, whereas $\neg[\text{the } x\colon Ax]\neg Bx$ is always true
  - self-predication principles: what is the logical status of ‘the A is A’?
    - $[\text{the } x\colon Ax]Ax$ is not a FOL-tautology
    - $\neg[\text{the } x\colon Ax]\neg Ax$ is a FOL-tautology

- the Aristotelian square for definite descriptions is not Boolean closed
- its Boolean closure is a JSB hexagon
- importance of the (EX)- and (UN)-conditions

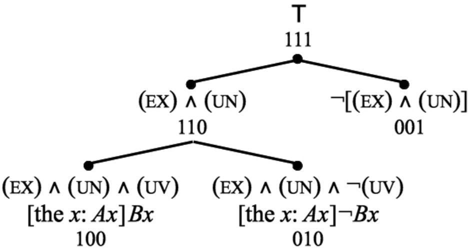




- consider the formulas in the definite descripton square/hexagon
- these formulas induce the partition Π^{FOL}_{TDD}:
  - $\alpha_1 := [\text{the } x\colon Ax]Bx$
  - $\alpha_2 := [\text{the } x\colon Ax]\neg Bx$
  - $\alpha_3 := \neg[(\textsc{ex}) \wedge (\textsc{un})]$
- example bitstring representations:
  - $[\text{the } x\colon Ax]Bx \equiv_{\mathsf{FOL}} \alpha_1$ $\rightsquigarrow$ gets represented as 100
  - $\neg[\text{the } x\colon Ax]\neg Bx \equiv_{\mathsf{FOL}} \alpha_1 \vee \alpha_3$ $\rightsquigarrow$ gets represented as 101
- logical perspective: the Boolean closure of the square/hexagon has $2^3 - 2 = 6$ contingent formulas
- conceptual/linguistic perspective: recursive partitioning of logical space


110
(EX) ∧ (UN)
[the x: Ax]Bx
100
[the x: Ax]¬Bx
010
101
¬[the x: Ax]¬Bx
011
¬[the x: Ax]Bx
¬[(EX) ∧ (UN)]
001

- view Π^{FOL}_{TDD} as the result of a process of recursively partitioning and restricting logical space (Seuren, Jaspers, Roelandt)
  - divide the logical universe: $(\textsc{ex}) \wedge (\textsc{un})$ vs. $\neg[(\textsc{ex}) \wedge (\textsc{un})]$
  - focus on the logical subuniverse defined by $(\textsc{ex}) \wedge (\textsc{un})$
  - recursively divide this subuniverse: $[\text{the } x\colon Ax]Bx$ vs. $[\text{the } x\colon Ax]\neg Bx$



- another look at the ambiguity pointed out by Russell
  - 'the A is B' unambiguously corresponds to $[\text{the } x\colon Ax]Bx = 100$
  - relative to the entire universe, its negation is $\neg[\text{the } x\colon Ax]Bx = 011$
  - relative to the subuniverse (**11**0), its negation is $[\text{the } x\colon Ax]\neg Bx = 010$

  $\Rightarrow$ Russell's two interpretations of 'the A is not B' correspond to negations of 'the A is B' *relative to two different universes* (semantic instead of syntactic characterization)

- Seuren and Jaspers's (2014) defeasible Principle of Complement Selection: "Natural complement selection is primarily relative to the proximate subuniverse, but there are overriding factors."

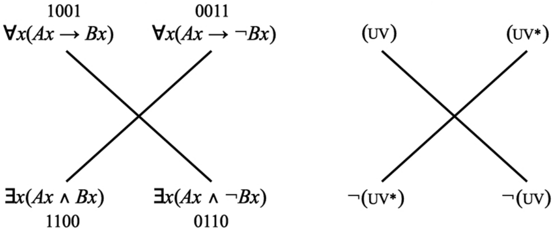
- the four categorical statements from syllogistics:

A	all As are B	$\forall x(Ax \rightarrow Bx)$
I	some As are B	$\exists x(Ax \wedge Bx)$
E	no As are B	$\forall x(Ax \rightarrow \neg Bx)$
O	some As are not B	$\exists x(Ax \wedge \neg Bx)$

- already implicit in the definite description formulas
  - $[\text{the } x\colon Ax]\ Bx \equiv_{\mathsf{FOL}} (\textsc{ex}) \wedge (\textsc{un}) \wedge (\textsc{uv})$
  - $\neg[\text{the } x\colon Ax]\ Bx \equiv_{\mathsf{FOL}} \neg(\textsc{ex}) \vee \neg(\textsc{un}) \vee \neg(\textsc{uv})$
  - $[\text{the } x\colon Ax]\neg Bx \equiv_{\mathsf{FOL}} (\textsc{ex}) \wedge (\textsc{un}) \wedge (\textsc{uv}^*)$
  - $\neg[\text{the } x\colon Ax]\neg Bx \equiv_{\mathsf{FOL}} \neg(\textsc{ex}) \vee \neg(\textsc{un}) \vee \neg(\textsc{uv}^*)$

$$(\textsc{uv}) \equiv_{\mathsf{FOL}} \forall x(Ax \rightarrow Bx) = \text{A}$$
$$\neg(\textsc{uv}) \equiv_{\mathsf{FOL}} \exists x(Ax \wedge \neg Bx) = \text{O}$$
$$(\textsc{uv}^*) \equiv_{\mathsf{FOL}} \forall x(Ax \rightarrow \neg Bx) = \text{E}$$
$$\neg(\textsc{uv}^*) \equiv_{\mathsf{FOL}} \exists x(Ax \wedge Bx) = \text{I}$$

- first-order logic (FOL) has no existential import
- the categorical statements induce the partition Π^{FOL}_{CAT}:
  - $\beta_1 := \exists x Ax \wedge \forall x(Ax \to Bx)$
  - $\beta_2 := \exists x(Ax \wedge Bx) \wedge \exists x(Ax \wedge \neg Bx)$
  - $\beta_3 := \exists x Ax \wedge \forall x(Ax \to \neg Bx)$
  - $\beta_4 := \neg\exists x Ax$ (recursive partitioning)
- the categorical statements constitute a degenerate square



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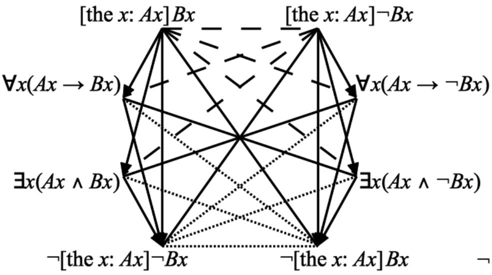
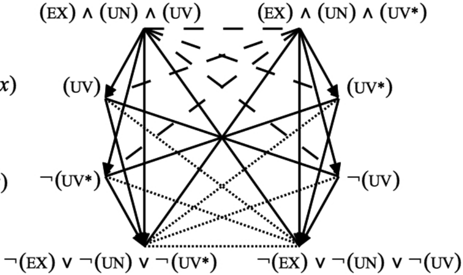
- there is a subalternation from [the x: Ax]Bx to the A-statement
  - FOL $\models [(\textsc{ex}) \wedge (\textsc{un}) \wedge (\textsc{uv})] \rightarrow (\textsc{uv})$
  - but not vice versa
- there is a subalternation from [the x: Ax]Bx to the I-statement
  - FOL $\models [(\textsc{ex}) \wedge (\textsc{uv})] \rightarrow \neg(\textsc{uv}^*)$
    so a fortiori FOL $\models [(\textsc{ex}) \wedge (\textsc{un}) \wedge (\textsc{uv})] \rightarrow \neg(\textsc{uv}^*)$
  - but not vice versa
- and so on...
- summary:

the interaction between the definite description formulas and the categorical statements gives rise a Buridan octagon


[the x: Ax]Bx
[the x: Ax]¬Bx
∀x(Ax → Bx)
∀x(Ax → ¬Bx)
∃x(Ax ∧ Bx)
∃x(Ax ∧ ¬Bx)
¬[the x: Ax]¬Bx
¬[the x: Ax]Bx


(EX) ∧ (UN) ∧ (UV)
(EX) ∧ (UN) ∧ (UV*)
(UV)
(UV*)
¬(UV*)
¬(UV)
¬(EX) ∨ ¬(UN) ∨ ¬(UV*)
¬(EX) ∨ ¬(UN) ∨ ¬(UV)

- the definite descriptions induce the partition $\Pi^{\mathsf{FOL}}_{\mathit{TDD}}$
- the categorical statements induce the partition $\Pi^{\mathsf{FOL}}_{\mathit{CAT}}$

  $\Rightarrow$ together, they induce the partition $\Pi^{\mathsf{FOL}}_{\mathit{OCTA}} = \Pi^{\mathsf{FOL}}_{\mathit{TDD}} \wedge_{\mathsf{FOL}} \Pi^{\mathsf{FOL}}_{\mathit{CAT}}$

  - $\gamma_1 := \exists x \exists y (Ax \wedge Ay \wedge x \neq y) \wedge \forall x (Ax \rightarrow Bx)$
  - $\gamma_2 := \exists x (Ax \wedge Bx) \wedge \exists x (Ax \wedge \neg Bx)$
  - $\gamma_3 := \exists x \exists y (Ax \wedge Ay \wedge x \neq y) \wedge \forall x (Ax \rightarrow \neg Bx)$
  - $\gamma_4 := [\text{the } x\colon Ax] Bx$
  - $\gamma_5 := [\text{the } x\colon Ax] \neg Bx$
  - $\gamma_6 := \neg \exists x Ax$

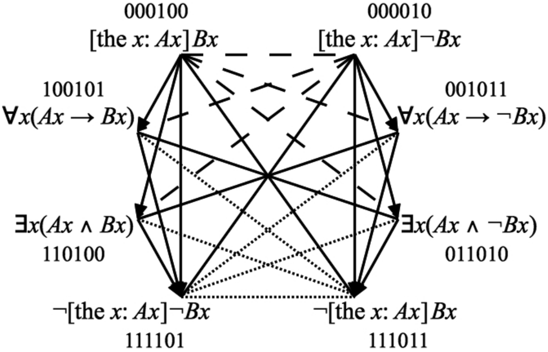
- $\Pi^{\mathsf{FOL}}_{\mathit{OCTA}}$ is a refinement of $\Pi^{\mathsf{FOL}}_{\mathit{TDD}}$

  $\Rightarrow$ $\gamma_4 = \alpha_1$ and $\gamma_5 = \alpha_2$, while $\gamma_1 \vee \gamma_2 \vee \gamma_3 \vee \gamma_6 \equiv_{\mathsf{FOL}} \alpha_3$

- $\Pi^{\mathsf{FOL}}_{\mathit{OCTA}}$ is a refinement of $\Pi^{\mathsf{FOL}}_{\mathit{CAT}}$

  $\Rightarrow$ $\gamma_2 = \beta_2$ and $\gamma_6 = \beta_4$, while $\gamma_1 \vee \gamma_4 \equiv_{\mathsf{FOL}} \beta_1$ and $\gamma_3 \vee \gamma_5 \equiv_{\mathsf{FOL}} \beta_3$

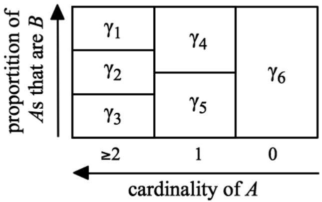
- $\Pi^{\mathsf{FOL}}_{\mathit{OCTA}}$ allows us to encode every formula of the Buridan octagon
- the Boolean closure of this octagon has $2^6 - 2 = 62$ contingent formulas



- $\Pi^{\mathsf{FOL}}_{\mathit{OCTA}}$ is ordered along two semi-independent dimensions
  - the cardinality of (the extension of) A
  - the proportion of As that are B

- *semi*-independent: higher cardinalities allow for more fine-grained proportionality distinctions

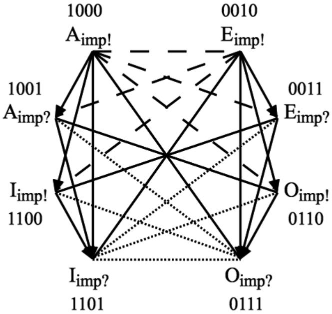
- ongoing work on linguistic aspects:
  - plausible partitioning process?
  - split the '≥ 2'-region into '≥ 3'- and '$= 2$'-subregions ('both', 'neither')

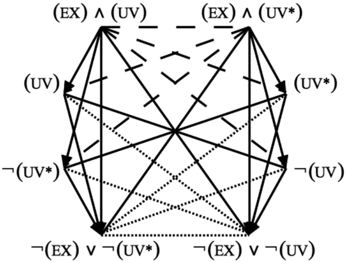


- recent work on existential import in syllogistics (Seuren, **Chatti and Schang**, Read)
- for every categorical statement φ, define
  - variant $\varphi_{\text{imp!}}$ that explicitly has existential import $\exists x Ax \wedge \varphi$
  - variant $\varphi_{\text{imp?}}$ that explicitly lacks existential import $\exists x Ax \rightarrow \varphi$

$\mathsf{A}_{\text{imp?}}$	$\equiv_{\text{FOL}}$	$\forall x(Ax \rightarrow Bx)$	$\equiv_{\text{FOL}}$	(UV)
$\mathsf{I}_{\text{imp!}}$	$\equiv_{\text{FOL}}$	$\exists x(Ax \wedge Bx)$	$\equiv_{\text{FOL}}$	$\neg(\text{UV}^*)$
$\mathsf{E}_{\text{imp?}}$	$\equiv_{\text{FOL}}$	$\forall x(Ax \rightarrow \neg Bx)$	$\equiv_{\text{FOL}}$	(UV^*)
$\mathsf{O}_{\text{imp!}}$	$\equiv_{\text{FOL}}$	$\exists x(Ax \wedge \neg Bx)$	$\equiv_{\text{FOL}}$	$\neg(\text{UV})$
$\mathsf{A}_{\text{imp!}}$	$\equiv_{\text{FOL}}$	$\exists x Ax \wedge \forall x(Ax \rightarrow Bx)$	$\equiv_{\text{FOL}}$	$(\text{EX}) \wedge (\text{UV})$
$\mathsf{I}_{\text{imp?}}$	$\equiv_{\text{FOL}}$	$\exists x Ax \rightarrow \exists x(Ax \wedge Bx)$	$\equiv_{\text{FOL}}$	$\neg(\text{EX}) \vee \neg(\text{UV}^*)$
$\mathsf{E}_{\text{imp!}}$	$\equiv_{\text{FOL}}$	$\exists x Ax \wedge \forall x(Ax \rightarrow \neg Bx)$	$\equiv_{\text{FOL}}$	$(\text{EX}) \wedge (\text{UV}^*)$
$\mathsf{O}_{\text{imp?}}$	$\equiv_{\text{FOL}}$	$\exists x Ax \rightarrow \exists x(Ax \wedge \neg Bx)$	$\equiv_{\text{FOL}}$	$\neg(\text{EX}) \vee \neg(\text{UV})$

- Chatti and Schang's 8 formulas are closely related to our 8 formulas
- Chatti and Schang's 8 formulas also constitute a Buridan octagon
- bitstring analysis: partition $\{\mathsf{A}_{\mathsf{imp!}}, \mathsf{I}_{\mathsf{imp!}} \wedge \mathsf{O}_{\mathsf{imp!}}, \mathsf{E}_{\mathsf{imp!}}, \neg\exists x Ax\} = \Pi^{\mathsf{FOL}}_{CAT}$





KU LEUVEN

A Related Octagon

- Buridan octagon for definite description formulas and categorical statements
  - induces the partition $\Pi^{\mathsf{FOL}}_{OCTA}$
  - its Boolean closure has $2^6 - 2 = 62$ formulas
  - $[\text{the } x\colon Ax]Bx \not\equiv_{\mathsf{FOL}} \mathsf{A} \wedge \mathsf{I}$ $(000100 \neq 100101 \wedge 110100)$

- Buridan octagon for categorical statements that explicitly have/lack existential import
  - induces the partition Π^{FOL}_{CAT}
  - its Boolean closure has $2^4 - 2 = 14$ formulas
  - $\mathsf{A}_{\mathsf{imp!}} \equiv_{\mathsf{FOL}} \mathsf{A}_{\mathsf{imp?}} \wedge \mathsf{I}_{\mathsf{imp!}}$ $(1000 = 1001 \wedge 1100)$

- summary:
  - one and the same Aristotelian type (Buridan)
  - different Boolean subtypes

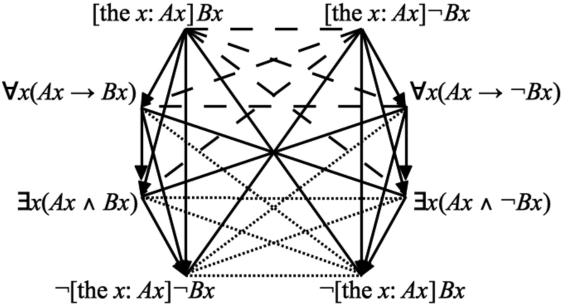
- until now: only worked in ordinary first-order logic (FOL)
- Chatti and Schang: deal with existential import by adding $(\neg)\exists x Ax$ as conjunct/disjunct to the categorical statements

- alternative approach:
  - existential import $\neq$ property of individual formulas
  - existential import $=$ property of underlying logical system

- introduce new logical system SYL:
  - SYL = FOL + $\exists x Ax$
  - interpreted on FOL-models $\langle D, I\rangle$ such that $I(A) \neq \emptyset$

  - analogy with modal logic:
    - D = K + $\Diamond\top$
    - interpreted on serial frames,
      i.e. K-frames $\langle W, R\rangle$ such that $R[w] \neq \emptyset$ (for all $w \in W$)

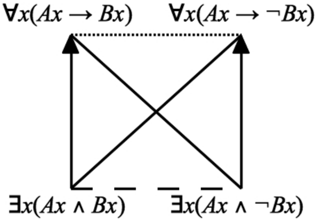
- move from FOL to SYL
- influence on the categorical statements:
  - e.g. A and E are independent in FOL, but become contrary in SYL, etc.
  - degenerate square turns into classical square
- no influence on the definite description formulas:
  - e.g. $[\text{the } x\colon Ax]Bx$ and $[\text{the } x\colon Ax]\neg Bx$ are contrary in FOL, and remain so in SYL
  - classical square remains classical square
- no influence on the interaction between definite descriptions and categorical statements:
  - e.g. subalternation from $[\text{the } x\colon Ax]Bx$ to A and I in FOL, and this remains so in SYL
- from Buridan octagon to Lenzen octagon

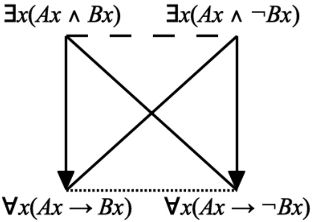


[the x: Ax]Bx
[the x: Ax]¬Bx
∀x(Ax → Bx)
∀x(Ax → ¬Bx)
∃x(Ax ∧ Bx)
∃x(Ax ∧ ¬Bx)
¬[the x: Ax]¬Bx
¬[the x: Ax]Bx

- which partition $\Pi^{\mathsf{SYL}}_{\mathit{OCTA}}$ is induced?
  - SYL is a stronger logical system than FOL
  - consider $\neg\exists x Ax = \gamma_6 \in \Pi^{\mathsf{SYL}}_{\mathit{OCTA}}$: FOL-consistent, but SYL-inconsistent
  - $\Pi^{\mathsf{SYL}}_{\mathit{OCTA}} = \Pi^{\mathsf{FOL}}_{\mathit{OCTA}} - \{\gamma_6\}$
- inverse correlation between axiomatic strength and Boolean complexity:
  - FOL $\rightsquigarrow$ Buridan octagon $\rightsquigarrow$ Boolean closure of $2^6 - 2 = 62$ contingencies
  - SYL $\rightsquigarrow$ Lenzen octagon $\rightsquigarrow$ Boolean closure of $2^5 - 2 = 30$ contingencies
- deleting the sixth bit position $\Rightarrow$ unified perspective on all changes:
  - A (100101) and E (001011) change from unconnected to contary
  - I (110100) and O (011010) change from unconnected to subcontrary
  - A (100101) and I (110100) change from unconnected to subaltern
  - $[$the x: $Ax]Bx$ (000100) and $[$the x: $Ax]Bx$ (000010) are contrary and remain so
  - $[$the x: $Ax]Bx$ (000100) and A (100101) are subaltern and remain so

- (EX) and (UN) play complementary roles in Russell's theory
- introduce new logical system SYL*
  - SYL* = FOL + $\forall x \forall y((Ax \wedge Ay) \rightarrow x = y)$
  - interpreted on FOL-models $\langle D, I\rangle$ such that $|I(A)| \leq 1$
- move from FOL to SYL*
- no influence on the definite description formulas
  - e.g. [the x: Ax]Bx and [the x: Ax]$\neg Bx$ are contrary in FOL, and remain so in SYL
  - classical square remains classical square
- influence on the categorical statements:
  - e.g. A and E are independent in FOL, but become subcontrary in SYL
  - degenerate square turns into classical square
  - note: this square is 'flipped upside down'!



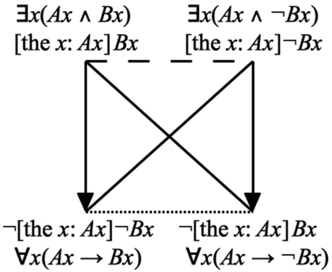


- example: take A to be the predicate 'king of country C'
- then always $|I(A)| \leq 1$
  - if C is a monarchy, then $|I(A)| = 1$
  - if C is a republic, then $|I(A)| = 0$

- move from FOL to SYL*
- influence on the interaction between definite descriptions and categorical statements
  - e.g. $[\text{the } x\colon Ax]Bx$ and the E-statement go from FOL-contrary to SYL*-contradictory
  - e.g. in FOL there is a subalternation from $[\text{the } x\colon Ax]Bx$ to the I-statement, but in SYL* they are logically equivalent to each other
- pairwise collapse of def. descr. formulas and categorical statements:

$$\begin{array}{rcccl} [\text{the } x\colon Ax]Bx & \equiv_{\mathsf{SYL}^*} & \mathsf{I} & = & \exists x(Ax \wedge Bx), \\ \neg[\text{the } x\colon Ax]Bx & \equiv_{\mathsf{SYL}^*} & \mathsf{E} & = & \forall x(Ax \rightarrow \neg Bx), \\ [\text{the } x\colon Ax]\neg Bx & \equiv_{\mathsf{SYL}^*} & \mathsf{O} & = & \exists x(Ax \wedge \neg Bx), \\ \neg[\text{the } x\colon Ax]\neg Bx & \equiv_{\mathsf{SYL}^*} & \mathsf{A} & = & \forall x(Ax \rightarrow Bx). \end{array}$$

- from Buridan octagon to collapsed (flipped) classical square



$\exists x(Ax \wedge Bx)$
[the x: Ax]Bx
$\exists x(Ax \wedge \neg Bx)$
[the x: Ax]$\neg Bx$
$\neg$[the x: Ax]$\neg Bx$
$\forall x(Ax \rightarrow Bx)$
$\neg$[the x: Ax]Bx
$\forall x(Ax \rightarrow \neg Bx)$

- elementary calculation yields the partition $\Pi^{\mathsf{SYL}^*}_{COLL}$ $= \{\exists x Ax \wedge \forall x(Ax \rightarrow Bx), \exists x Ax \wedge \forall x(Ax \rightarrow \neg Bx), \neg \exists x Ax\}$
- $\Pi^{\mathsf{SYL}^*}_{COLL} = \Pi^{\mathsf{FOL}}_{OCTA} - \{\gamma_1, \gamma_2, \gamma_3\}$
  - SYL* is a stronger logical system than FOL
  - $\gamma_1, \gamma_2, \gamma_3$ are FOL-consistent, but SYL*-inconsistent
- $\Pi^{\mathsf{SYL}^*}_{COLL} = \Pi^{\mathsf{FOL}}_{TDD}$
  - Π^{FOL}_{TDD} is the partition for the def. descr. square in FOL
  - moving from FOL to SYL* did not change this square
  - but did cause it to coincide with the categorical statement square
- $\Pi^{\mathsf{SYL}^*}_{COLL} = \Pi^{\mathsf{FOL}}_{CAT} - \{\beta_2\}$
  - Π^{FOL}_{CAT} is the partition for the cat. statement square in FOL
  - SYL* is a stronger than FOL; β_2 is FOL-consistent, but SYL*-inconsistent
  - moving from FOL to SYL* triggered change from degen. square to (flipped) classical square, which coincides with the def. descr. square

Connection with PAL

- the categorical statements yield a flipped classical square in SYL*
  $\Rightarrow$ quantification over a domain of at most one element $(|I(A)| \leq 1)$

- similar situation in public announcement logic (PAL) (Demey 2012)

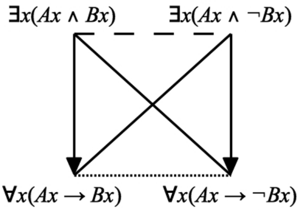
- standard semantics: model update operation $(\mathbb{M}, w) \mapsto (\mathbb{M}^\varphi, w^\varphi)$

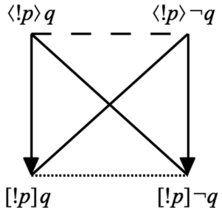
$$(\mathbb{M}, w) \models [!\varphi]\psi \quad \text{iff} \quad \text{if } (\mathbb{M}, w) \models \varphi \text{ then } (\mathbb{M}^\varphi, w^\varphi) \models \psi,$$
$$(\mathbb{M}, w) \models \langle !\varphi\rangle\psi \quad \text{iff} \quad (\mathbb{M}, w) \models \varphi \text{ and } (\mathbb{M}^\varphi, w^\varphi) \models \psi.$$

- informal quantificational interpretation:

  $[!\varphi]\psi$ iff after all public announcements of φ, it holds that ψ
  $[!\varphi]\psi$ iff after at least one public ann. of φ, it holds that ψ

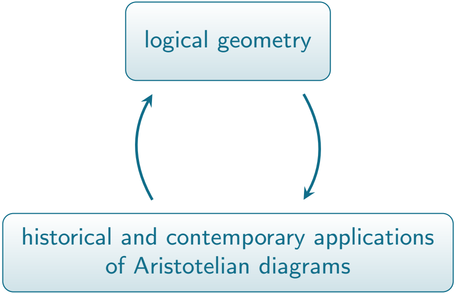
- informal quantificational interpretation: $[!\varphi]$ and $\langle !\varphi\rangle$ as universal/existential quantifiers over the set of public ann. of φ
- since $(\mathbb{M}, w) \mapsto (\mathbb{M}^\varphi, w^\varphi)$ is a partial function, the set of all public announcements of φ contains at most one element
  - if $(\mathbb{M}, w) \models \varphi$, then $(\mathbb{M}^\varphi, w^\varphi)$ is uniquely defined, i.e. there is exactly one public announcement of φ
  - if $(\mathbb{M}, w) \not\models \varphi$, then $(\mathbb{M}^\varphi, w^\varphi)$ is undefined, i.e. there is no public announcement of φ





Conclusion

- Aristotelian diagrams for Russell's theory of definite descriptions
  - classical square, JSB hexagon, Buridan octagon. . .
  - the formula $\neg$[the x: Ax]$\neg Bx$ and its interpretation, negations of [the x: Ax]Bx relative to different subuniverses. . .
- phenomena and techniques studied in logical geometry
  - bitstring analysis, Boolean closure. . .
  - Boolean subtypes, logic-sensitivity. . .



Thank you!

More info: www.logicalgeometry.org